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Sadržaj

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Monoids, Segal's condition and bisimplicial spaces

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Abstract

A characterization of simplicial objects in categories with finite products obtained by the reduced bar construction is given. The condition that characterizes such simplicial objects is a strictification of Segal's condition guaranteeing that the loop space of the geometric realization of a simplicial space X and the space X_1 are of the same homotopy type. A generalization of Segal's result appropriate for bisimplicial spaces is given. This generalization gives conditions guaranteing that the double loop space of the geometric realization of a bisimplicial space X and the space X_{11} are of the same homotopy type.

Mathematics Subject Classification (2010): 18G30, 57T30, 55P35

Keywords: reduced bar construction, simplicial space, loop space

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1 Introduction

This paper is based on the author talks delivered in 2014 at the Fourth Mathematical Conference of the Republic of Srpska and at the CGTA Colloquium of the Faculty of Mathematics in Belgrade. Its first part gives a condition which is necessary and sufficient for a simplicial object to be obtained by the reduced bar construction. It turns out that this condition is a strictification of Segal's condition guaranteeing that the loop space of the geometric realization of a simplicial space X and the space X_1 are of the same homotopy type.

The second part of this paper is devoted to a generalization of Segal's result. This generalization gives conditions guaranteing that the double loop space of the geometric realization of a bisimplicial space X and the space X_{11} are of the same homotopy type. We refer to [8] for a complete generalization of Segal's result.

2 Monoids and the reduced bar construction

A *strict monoidal* category $(\mathcal{M}, \otimes, \mathcal{I})$ is a category $\mathcal{M}$ with an associative bifunctor $\otimes\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$,

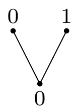
$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad \text{and} \quad (f \otimes g) \otimes h = f \otimes (g \otimes h),$$

and an object I, which is a left and right unit for $\otimes$,

$$A \otimes I = A = I \otimes A \quad \text{and} \quad f \otimes \mathbf{1}_I = f = \mathbf{1}_I \otimes f.$$

A *strict monoidal functor* between strict monoidal categories is a functor that preserves strict monoidal structure "on the nose", i.e., $F(A \otimes B) = F(A) \otimes F(B)$, $F(I) = I$, etc.

Algebraist's *simplicial category* Δ is an example of strict monoidal category. The objects of Δ are all finite ordinals $0 = \emptyset$, $1 = \{0\}$, $\ldots, n = \{0, \ldots, n-1\}$, etc. The arrows of Δ from n to m are all order preserving functions from the set n to the set m, i.e., $f : n \to m$ satisfying: if $i < j$ and $i, j \in n$, then $f(i) \leq f(j)$. We use the standard graphical presentation for arrows of Δ. For example, the unique arrows from 2 to 1 and from 0 to 1 are graphically presented as:

$2 \to 1$ 

$0 \to 1$ 

A bifunctor $\otimes : \Delta \times \Delta \to \Delta$ is defined on objects as addition and on arrows as placing "side by side", i.e., for $f : n \to m$ and $f' : n' \to m'$

$$(f \otimes f')(i) = \begin{cases} f(i), & \text{when} \quad 0 \leq i \leq n-1, \\ m + f'(i-n), & \text{when} \quad n \leq i \leq n + n' - 1, \end{cases}$$

and 0 serves as the unit I.

A *monoid* in a strict monoidal category $(\mathcal{M}, \otimes, I)$ is a triple $(M, \mu : M \otimes M \to M, \eta : I \to M)$ such that

$$\mu \circ (\mu \otimes \mathbf{1}_M) = \mu \circ (\mathbf{1}_M \otimes \mu) \quad \text{and} \quad \mu \circ (\mathbf{1}_M \otimes \eta) = \mathbf{1}_M = \mu \circ (\eta \otimes \mathbf{1}_M).$$

For example, $(1, \bigvee, \cdot)$ is a monoid in Δ, where $\bigvee$ and $\cdot$ are the above graphical presentations of the arrows of Δ from 2 to 1, and from 0 to 1. The following result, taken over from [3, VII.5, Proposition 1], shows the "universal" property of this monoid.

Proposition 2.1. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$, there is a unique strict monoidal functor $F : \Delta \to \mathcal{M}$ such that $F(1) = M$, $F(\bigvee) = \mu$ and $F(\cdot) = \eta$.*

Let Δ_{par} be the category with the same objects as Δ, whose arrows are order preserving partial functions. Then $(1, \bigvee, \cdot)$ is a monoid in the strict monoidal category Δ_{par} with the same tensor and unit as Δ. The empty partial function from 1 to 0 is graphically presented as $\dot{\ }$. By [7, Proposition 6.2] we have the following universal property of this monoid.

Proposition 2.2. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, there is a unique strict monoidal functor $F : \Delta_{par} \to \mathcal{M}$ such that $F(1) = M$, $F(\bigvee) = \mu$, $F(\,.\,) = \eta$ and $F(\dot{\ })$ is the unique arrow from M to the unit (a terminal object of $\mathcal{M}$).*

Topologist's simplicial category is the full subcategory of Δ on nonempty ordinals as objects. We identify this category with the subcategory of *Top*. The object $n+1$ is identified with the standard ordered simplex

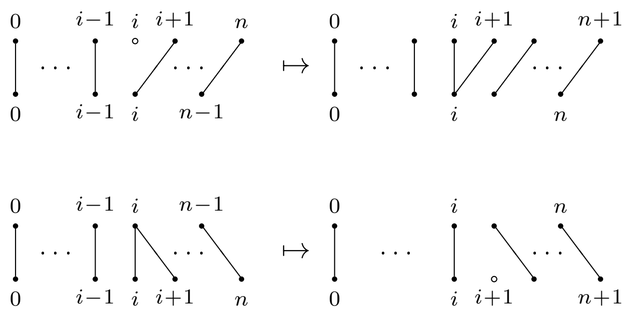
$$\Delta^n = \{(t_0, \dots, t_n) \mid t_0, \dots, t_n \geq 0, \sum_i t_i = 1\},$$

and an arrow $f : n+1 \to m+1$ is identified with the affine map defined by

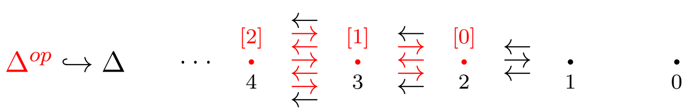
$$f(t_0, \dots, t_n) = (s_0, \dots, s_m), \text{ where } s_j = \sum_{f(i)=j} t_i.$$

We denote by Δ^{op} the opposite of topologist's simplicial category and rename its objects so that the ordinal $n+1$ is denoted by $[n]$, i.e., $[n] = \{0, \dots, n\}$. Let Δ_{Int} be the subcategory of Δ whose objects are finite ordinals greater or equal to 2 and whose arrows are interval maps, i.e., order-preserving functions, which preserve, moreover, the first and the last element.

The categories Δ^{op} and Δ_{Int} are isomorphic by the functor $\mathcal{J}$ (see [7, Section 6]). This functor maps the object $[n]$ to $n+2$ and it maps the generating arrows of Δ^{op} in the following way.

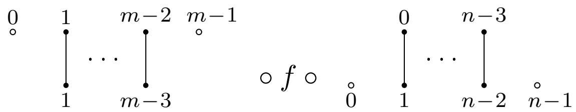


The functor $\mathcal{J}$ may be visualized as the following embedding of Δ^{op} into Δ. (I am grateful to Matija Bašić for this remark.)


(1)

Throughout this paper, we represent the arrows of Δ^{op} by the graphical presentations of their $\mathcal{J}$ images in Δ_{Int}.

We have a functor $\mathcal{H}\colon \Delta_{Int} \to \Delta_{par}$ defined on objects as $\mathcal{H}(n) = n-2$, and on arrows, for $f\colon n \to m$, as

$\mathcal{H}(f) =$ 

(Intuitively, $\mathcal{H}(f)$ is obtained by omitting the vertices 0, $n-1$ from the source, and 0, $m-1$ from the target in the graphical presentation of f together with all the edges incident to these vertices.) It is not difficult to see that $\mathcal{H}(\mathbf{1}_n) = \mathbf{1}_{n-2}$, and that for a pair of arrows $f\colon n \to m$ and $g\colon m \to k$ of Δ_{Int} we have

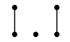
$$\mathcal{H}(g)\circ\mathcal{H}(f)(i) = \begin{cases} g(f(i+1)) - 1, & f(i+1) \notin \{0, m-1\} \wedge g(f(i+1)) \notin \{0, k-1\} \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and

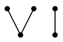
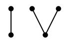
$$\mathcal{H}(g\circ f)(i) = \begin{cases} g(f(i+1)) - 1, & g(f(i+1)) \notin \{0, k-1\} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Since $g(f(i+1)) \notin \{0, k-1\}$ implies $f(i+1) \notin \{0, m-1\}$, we have that $\mathcal{H}(g) \circ \mathcal{H}(f)(i) = \mathcal{H}(g \circ f)(i)$, and $\mathcal{H}$ so defined is indeed a functor.

A *simplicial object* X in a category $\mathcal{M}$ is a functor $X : \Delta^{op} \to \mathcal{M}$. The following proposition is a corollary of Proposition 2.2.

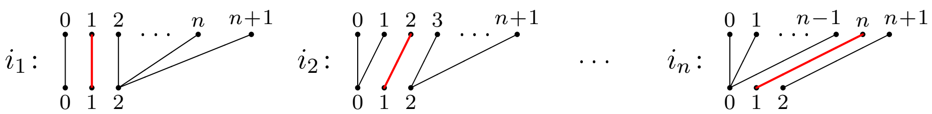
Proposition 2.3. *Given a monoid (M, μ, η) in a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, there is a simplicial object X in $\mathcal{M}$ such that $X([n]) = M^n$, $X($**$) = \mu$, $X($**$) = \eta$.*

Proof. Take X to be the composition $F \circ \mathcal{H} \circ \mathcal{J}$, for F as in Proposition 2.2. ⊣

Note that both  and  are mapped by X to the unique arrow from M to the unit M^0 (a terminal object of $\mathcal{M}$). We say that a simplicial object in $\mathcal{M}$ obtained as the composition $F \circ \mathcal{H} \circ \mathcal{J}$, for F as in Proposition 2.2, is the *reduced bar construction based* on M (see [10] and [7]).

For X a simplicial object, we abbreviate $X([n])$ by X_n. Also, for f an arrow of Δ^{op}, we abbreviate $X(f)$ by f whenever the simplicial object X is determined by the context.

For $n \geq 2$, consider the arrows $i_1, \ldots, i_n : [n] \to [1]$ of Δ^{op} graphically presented as follows.



(It would be more appropriate to denote these arrows by $i_1^n, \ldots, i_n^n$, but we omit the upper indices taking them for granted.)

For arrows $f\colon C \to A$ and $g\colon C \to B$ of a strict monoidal category $\mathcal{M}$ whose monoidal structure is given by finite products, we denote by $\langle f, g\rangle : C \to A \times B$ the arrow obtained by the universal property of product in $\mathcal{M}$. For X a simplicial object in $\mathcal{M}$, we denote by p_0 the unique arrow from X_0 to the unit, i.e., a terminal object $(X_1)^0$ of $\mathcal{M}$, and we denote by p_1 the identity arrow from X_1 to X_1. For $n \geq 2$ and the above mentioned arrows $i_1, \ldots, i_n : [n] \to [1]$ of Δ^{op}, we denote by p_n the arrow

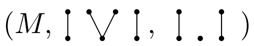
$$\langle i_1, \ldots, i_n\rangle \colon X_n \to (X_1)^n,$$

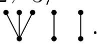
where by our convention, i_j abbreviates $X(i_j)$.

Let X be the reduced bar construction based on a monoid M. Since X_0 is the unit M^0 and for $n \geq 2$, the arrow $i_j : M^n \to M$ is the jth projection, we have that for every $n \geq 0$, the arrow p_n is the identity. We show that this property characterizes the reduced bar construction based on a monoid in $\mathcal{M}$.

Proposition 2.4. *Let $\mathcal{M}$ be a strict monoidal category whose monoidal structure is given by finite products. A simplicial object X in $\mathcal{M}$ is the reduced bar construction based on a monoid in $\mathcal{M}$ if and only if for every $n \geq 0$, the arrow $p_n\colon X_n \to (X_1)^n$ is the identity.*

Proof. The "only if" part of the proof is given in the paragraph preceding this proposition. For the "if" part of the proof, let us denote X_1 by M. By our convention, the X images of arrows of Δ^{op} are denoted just by their names or graphical presentations. We show that

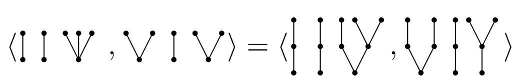
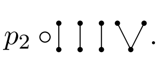


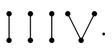
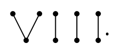
is a monoid in $\mathcal{M}$. Let $k^1_{M^2,M} : M^2 \times M \to M^2$ and $k^2_{M^2,M} : M^2 \times M \to M$ be the first and the second projection respectively. Since $p_3 = \langle i_1, i_2, i_3\rangle \colon M^3 \to M^3$ is the identity, we have that $k^1_{M^2,M} = \langle i_1, i_2\rangle$ and $k^2_{M^2,M} = i_3 =$ .

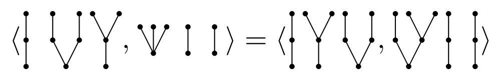
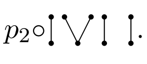
For arrows $f : C \to A$, $g : C \to B$, $h : D \to C$, $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ and projections $k^1_{A_1,A_2} \colon A_1 \times A_2 \to A_1$ and $k^2_{A_1,A_2} \colon A_1 \times A_2 \to A_2$, the following equations hold in $\mathcal{M}$

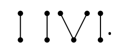
$$\langle f \circ h, g \circ h\rangle = \langle f, g\rangle \circ h, \qquad f_1 \times f_2 = \langle f_1 \circ k^1_{A_1,A_2}, f_2 \circ k^2_{A_1,A_2}\rangle.$$

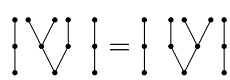
We have

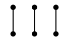
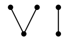
$k^1_{M^2,M} = \langle i_1, i_2\rangle =$  $= p_2 \circ$ .

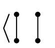
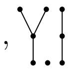
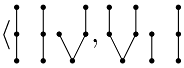
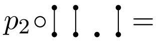
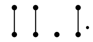
Hence, $k^1_{M^2,M} =$ . Analogously, we prove that $k^2_{M,M^2} =$ . Also,

$\mu \times \mathbf{1} = \langle \mu \circ k^1_{M^2,M}, k^2_{M^2,M}\rangle =$  $= p_2 \circ$ .

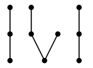
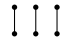
Hence, $\mu \times \mathbf{1} =$ . Analogously, we prove that $\mathbf{1} \times \mu =$ . Now, $\mu \circ (\mu \times \mathbf{1}) = \mu \circ (\mathbf{1} \times \mu)$, since



That $k^1_{M,M^0} = \mathbf{1} =$ , and $k^2_{M,M^0} =$  follows from the fact that M^0 is the strict unit and a terminal object of $\mathcal{M}$. Hence,

$\mathbf{1}\times\eta = \langle k^1_{M,M^0}, \eta\circ k^2_{M,M^0}\rangle = \langle$ ,  $\rangle = \langle$  $\rangle = p_2\circ$  $=$ .

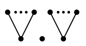
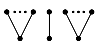
Now, $\mu \circ (\mathbf{1} \times \eta) = \mathbf{1}$, since

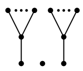
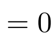
 $=$  $= \mathbf{1}.$

Analogously, we prove that $\mu \circ (\eta \times \mathbf{1}) = \mathbf{1}$, and conclude that M is a monoid in $\mathcal{M}$.

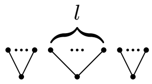
Let Y be the reduced bar construction based on M. We show that $X = Y$. It is clear that the object parts of the functors X and Y coincide. We prove that for every arrow $f\colon [m] \to [n]$ of Δ^{op}, the arrows $X(f)$ and $Y(f)$ are equal in $\mathcal{M}$.

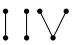
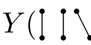
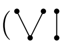
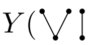
If $n = 0$, then this is trivial since X_0, which is equal to M^0, is a terminal object of $\mathcal{M}$. If $n = 1$, then f has one of the following forms

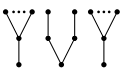
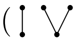
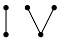
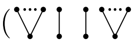
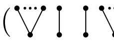
 or  or 

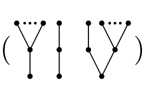
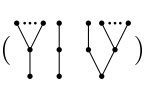
In the first case, $f =$  and the X and Y images of the upper part are equal as in the case $n = 0$, while $X($  $) = Y($  $)$ by the definition of Y.

In the second case, f is either identity and $X(f) = Y(f)$ holds, or f is i_j for some $1 \leq j \leq m$. From $\langle X(i_1), \ldots, X(i_m)\rangle = \mathbf{1}$, we conclude that $X(i_j)$ is the jth projection from M^m to M. On the other hand, by the definition of Y, we have that $Y(i_j)$ is the jth projection from M^m to M. Hence $X(f) = Y(f)$.

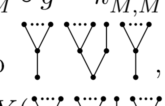
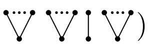
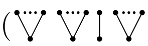
In the third case, when f is , we proceed by induction on $l \geq 2$. In the proof we use the fact that two arrows $g, h\colon C \to M^2$ are equal in $\mathcal{M}$ iff $k^1_{M,M} \circ g = k^1_{M,M} \circ h$ and $k^2_{M,M} \circ g = k^2_{M,M} \circ h$, where $k^1_{M,M}$ and $k^2_{M,M}$ are the first and the second projection from M^2 to M. Also, we know from above that

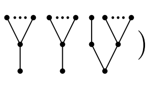
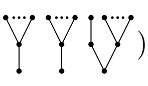
$k^1_{M,M} = X($  $) = Y($  $),\quad k^2_{M,M} = X($  $) = Y($  $).$

If $l = 2$, then f is equal to . Since $X($  $) = Y($  $)$, in order to prove that $X(f) = Y(f)$, it suffices to prove that $g = X($  $)$ is equal to $h = Y($  $)$. By relying on the second case for †, we have that

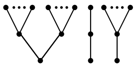
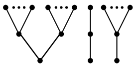
$k^1_{M,M} \circ g = X($  $) \overset{\dagger}{=} Y($  $) = k^1_{M,M} \circ h.$

Analogously, we prove that $k^2_{M,M} \circ g = k^2_{M,M} \circ h$. Hence, $g = h$.

If $l > 2$, then f is equal to , and it suffices to prove that $g = X($  $)$ is equal to $h = Y($  $)$. By relying on the induction hypothesis for †, we have that

$k^1_{M,M} \circ g = X($  $) \overset{\dagger}{=} Y($  $) = k^1_{M,M} \circ h.$

By relying on the second case for †, we have that

$k^2_{M,M} \circ g = X($  $) \overset{\dagger}{=} Y($  $) = k^2_{M,M} \circ h.$

Hence, $g = h$. This concludes the case when f maps $[m]$ to $[1]$.

Suppose now that $f\colon [m] \to [n]$ is an arrow of Δ^{op} and $n \geq 2$. As in the case when $n = 1$, we conclude that for every $1 \leq j \leq n$,

$$X(i_j \circ f) = Y(i_j \circ f).$$

Since,

$$\langle X(i_1), \ldots, X(i_n)\rangle = \langle Y(i_1), \ldots, Y(i_n)\rangle = \mathbf{1}_{M^n},$$

we have that

$$\begin{aligned} X(f) &= \langle X(i_1), \ldots, X(i_n)\rangle \circ X(f) = \langle X(i_1) \circ X(f), \ldots, X(i_n) \circ X(f)\rangle \\ &= \langle Y(i_1) \circ Y(f), \ldots, Y(i_n) \circ Y(f)\rangle = \langle Y(i_1), \ldots, Y(i_n)\rangle \circ Y(f) \\ &= Y(f). \end{aligned}$$

⊣

3 Segal's simplicial spaces

Let *Top* be the category of compactly generated Hausdorff spaces. For a simplicial object in *Top*, i.e., a *simplicial space* X, a relaxed form of the condition

for every n, $p_n\colon X_n \to (X_1)^n$ is the identity,

reads

for every n, $p_n\colon X_n \to (X_1)^n$ is a homotopy equivalence.

Segal, [9], used simplicial spaces satisfying this relaxed condition for his delooping constructions and we call them *Segal's simplicial spaces*. (Note that, for the sake of simplicity, this notion is weaker than the one defined in [8].) Essentially as in the proof of Proposition 2.4, one can show the following.

Proposition 3.1. *If $X : \Delta^{op} \to Top$ is Segal's simplicial space, then X_1 is a homotopy associative* H*-space whose multiplication is given by the composition*

$$(X_1)^2 \xrightarrow{p_2^{-1}} X_2 \xrightarrow{d_1^2} X_1,$$

where p_2^{-1} is an arbitrary homotopy inverse to p_2, and whose unit is $s_0^1(x_0)$, for an arbitrary $x_0 \in X_0$.

(A complete proof of this proposition is given in [8, Appendix, Proof of Lemma 3.1].)

The *realization* of a simplicial space X, is the quotient space

$$|X| = \left(\coprod_n X_n \times \Delta^n\right) \Big/ \sim,$$

where $\sim$ is the smallest equivalence relation on $\coprod_n X_n \times \Delta^n$ such that for every $f\colon [n] \to [m]$ of Δ, $x \in X_m$ and $t \in \Delta^n$

$$(f^{op}(x), t) \sim (x, f(t)).$$

A *simplicial map* is a natural transformation between simplicial spaces. Note that the realization is *functorial*, i.e., it is defined also for simplicial maps. For simplicial spaces X and Y the *product* $X \times Y$ is defined so that $(X \times Y)_n = X_n \times Y_n$ and $(X \times Y)(f) = X(f) \times Y(f)$. The nth component of the first projection $k^1\colon X \times Y \to X$ is the first projection $k^1_n : X_n \times Y_n \to X_n$ and analogously for the second projection. The realization functor preserves products of simplicial spaces (see [4, Theorem 14.3], [2, III.3, Theorem] and [5, Corollary 11.6]) in the sense that

$$\langle |k^1|, |k^2| \rangle \colon |X \times Y| \to |X| \times |Y|$$

is a homeomorphism.

The following two propositions stem from [9, Proposition 1.5 (b)] and from [6, Appendix, Theorem A4 (ii)] (see also [8, Lemma 2.11]).

Proposition 3.2. *Let $X : \Delta^{op} \to Top$ be Segal's simplicial space such that for every m, X_m is a CW-complex. If X_1 with respect to the* H*-space structure is grouplike, then $X_1 \simeq \Omega|X|$.*

Proposition 3.3. *Let $f : X \to Y$ be a simplicial map of simplicial spaces such that for every m, X_m and Y_m are CW-complexes. If each $f_m : X_m \to Y_m$ is a homotopy equivalence, then $|f|\colon |X| \to |Y|$ is a homotopy equivalence.*

4 Segal's bisimplicial spaces

A *bisimplicial space* is a functor $X\colon \Delta^{op} \times \Delta^{op} \to Top$ and it may be visualized as the following graph (see the red subgraph of (1)).

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots \\
\cdots & X_{22} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{12} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{02} \\
 & \downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow \\
\cdots & X_{21} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{11} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{01} \\
 & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\
\cdots & X_{20} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} & X_{10} & \substack{\rightarrow\\ \leftarrow\\ \rightarrow} & X_{00}
\end{array}$$

Let Y_n, for $n \geq 0$, be the realization of the nth column, i.e., $Y_n = |X_{n\_}|$. Since the realization is functorial, we obtain the simplicial space Y.

$$\cdots \quad Y_2 \quad \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \quad Y_1 \quad \substack{\rightarrow\\ \leftarrow\\ \rightarrow} \quad Y_0$$

The *realization* $|X|$ of the bisimplicial space X is the realization $|Y|$ of the simplicial space Y.

If the simplicial space $X_{1\_}$ is Segal's, then, by Proposition 3.1, X_{11} is a homotopy associative H-space and this is the H-space structure we refer to in the following proposition.

Proposition 4.1. *If $X\colon \Delta^{op} \times \Delta^{op} \to Top$ is a bisimplicial space such that $X_{1\_}$ is Segal's, X_{11} with respect to the* H*-space structure is grouplike, for every $m \geq 0$, $X_{\_m}$ is Segal's, and for every $n, m \geq 0$, X_{nm} and Y_n are CW-complexes, then $X_{11} \simeq \Omega^2|X|$.*

Proof. Since $X_{1\_}$ is Segal's simplicial space such that for every m, X_{1m} is a CW-complex and X_{11} with respect to the H-space structure is grouplike, by Proposition 3.2 we have that $X_{11} \simeq \Omega|X_{1\_}| = \Omega Y_1$.

For every m, $X_{\_m}$ is Segal's. Hence, for every n, the map $p_{nm} : X_{nm} \to (X_{1m})^n$, is a homotopy equivalence. The map p_{0m} is the unique map from X_{0m} to $(X_{1m})^0$, the map p_{1m} is the identity on X_{1m}, and for $n \geq 2$, the map p_{nm} is

$$\langle (i_1, m), \ldots, (i_n, m)\rangle \colon X_{nm} \to (X_{1m})^n.$$

Also, for every $f\colon [m] \to [m']$ of Δ^{op} and every n the following diagram commutes:

$$\begin{array}{ccc} X_{nm} & \xrightarrow{\quad p_{nm} \quad} & (X_{1m})^n \\ {\scriptstyle (n,f)}\downarrow & & \downarrow{\scriptstyle (1,f)^n} \\ X_{nm'} & \xrightarrow{\quad p_{nm'} \quad} & (X_{1m'})^n \end{array}$$

Hence, for every n, $p_{n\_}$ is a simplicial map.

$$\begin{array}{ccc} \vdots & & \vdots \\ X_{n2} & \xrightarrow{\quad p_{n2} \quad} & (X_{12})^n \\ \downarrow\uparrow\downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow\uparrow\downarrow \\ X_{n1} & \xrightarrow{\quad p_{n1} \quad} & (X_{11})^n \\ \downarrow\uparrow\downarrow & & \downarrow\uparrow\downarrow \\ X_{n0} & \xrightarrow{\quad p_{n0} \quad} & (X_{10})^n \end{array}$$

Every $(X_{1m})^n$ is a CW-complex since the product of CW-complexes in *Top* is a CW-complex. By Proposition 3.3, for every n, $|p_{n\_}| : Y_n \to |(X_{1\_})^n|$ is a homotopy equivalence. Since $|(X_{1\_})^0|$ is a singleton it is homeomorphic to $(Y_1)^0$ and we have that $p_0 : Y_0 \to (Y_1)^0$, as a composition of a homeomorphism with $|p_{0\_}|$, is a homotopy equivalence. The map $p_1\colon Y_1 \to Y_1$ is the identity. For $n \geq 2$, $\langle |k^1|, \ldots, |k^n|\rangle : |(X_{1\_})^n| \to |X_{1\_}|^n$ is a homeomorphism and for $1 \leq j \leq n$, $|(i_j, \_)| = |X(i_j, \_)| = Y(i_j)$. Hence, the map

$$p_n = \langle Y(i_1), \ldots, Y(i_n)\rangle = \langle |k^1|, \ldots, |k^n|\rangle \circ |\langle (i_1, \_), \ldots, (i_n, \_)\rangle|,$$

as a composition of a homeomorphism with $|p_{n\_}|$, is a homotopy equivalence between Y_n and $(Y_1)^n$. We conclude that Y is Segal's, and by Proposition 3.1, Y_1 is a homotopy associative H-space.

If a simplicial space is Segal's, then its realization is path-connected. This is because its value at [0] is contractible and therefore path-connected (see [5, Lemma 11.11]). Since $X_{1\_}$ is Segal's, we conclude that Y_1 is path-connected. Moreover, it is grouplike since every path-connected homotopy associative H-space, which is a CW-complex, is grouplike (see [1, Proposition 8.4.4]).

Applying Proposition 3.2 to Y, we obtain that $Y_1 \simeq \Omega|Y|$. Hence,

$$X_{11} \simeq \Omega Y_1 \simeq \Omega(\Omega|Y|) = \Omega^2|X|. \qquad \dashv$$

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The Remarks on Best and Coupled Best Approximations

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Abstract

In this paper we present some recent results on the best and coupled best approximations theorems in normed spaces.

1 Introduction

In this paper, we prove the existence of a solution for best approximations problem: "for set-valued maps F, G and H and set K, find $x_0 \in K$ such that

$$d(G(x_0), F(x_0)) \leq d(G(x), F(x_0)) \text{ for all } x \in K,\text{"}$$

and coupled best approximations problem: "find $(x_0, y_0) \in K \times K$ such that

$$d(G(x_0), F(x_0, y_0)) + d(H(y_0), F(y_0, x_0)) \leq d(G(x), F(x_0, y_0)) + d(H(y), F(y_0, x_0))$$

for all $(x, y) \in K \times K$", see [1]- [4], [12], [16], [17], [19], [20], [22], [26].

Let K be a given nonempty set, $f : K \times K \to \mathbb{R}$ a given map. The scalar equilibrium problem is: "find $x_0 \in K$ such that

$$f(x_0, y) \geq 0, \text{ for all } y \in K.\text{"} \tag{1}$$

Is well known that Problem (1) is a unified model of several problems, such as, variational inequality problems, saddle point problems, optimization problems and fixed point problems [11]. In 1972, Ky Fan [14] established his famous theorem, which interesting extensions have been given by several authors and a variety of applications, mostly in fixed point theory and approximation theory, have also given by many authors , see [5], [7], [13], [15], [18], [21] and references therein. Using the methods of the KKM-theory, we prove some results on scalar equilibrium problem in complete linear space. As corollaries, some results on the best approximations and coupled best approximations are obtained.

2 Preliminaries

We need the following definitions and results.

Let X and Y be non-empty sets and $F : X \multimap Y$ be a multimap (map) from a set X into the power set of a set Y. For $A \subseteq X$, let $F(A) = \cup\{F(x) : x \in A\}$. For any $B \subseteq Y$, the lower inverse and upper inverse of B under F are defined by

$$F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\} \text{ and } F^{+}(B) = \{x \in X : F(x) \subseteq B\},$$

respectively.

A map $F : X \multimap Y$ is upper (lower) semicontinuous on X if and only if for every open $V \subseteq Y$, the set $F^{+}(V)$ ($F^{-}(V)$) is open. A map $F : X \multimap Y$ is continuous if and only if it is upper and lower semicontinuous. A map $F : X \multimap Y$ with compact values is continuous if and only if F is a continuous map in the Hausdorff distance, see for example [9].

Let X be a normed space. If A and B are nonempty subsets of X and $\alpha, \beta \in \mathbb{R}$, we define

$$\alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B\} \text{ and } ||A|| = \inf\{||a|| : a \in A\}.$$

For a nonempty subset A of vector space X, let $\operatorname{conv}(A)$ denote the convex hull of A.

Definition 2.1. (Borwein, [10]) Let X and Y be real vector spaces, K a nonempty convex subset of X and C is a cone in Y. A map $F : K \multimap Y$ is said to be C-convex if,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) + C \tag{2}$$

for all $x_1, x_2 \in K$ and all $\lambda \in [0,1]$. A map F is convex if it satisfies condition (2) with $C = \{0\}$.

If F is a single-valued map, $Y = \mathbb{R}$ and $C = [0, +\infty)$, we obtain the standard definition of convex maps. The convex map play an important role in convex analysis, economic theory and convex optimization problems see for example [6], [8], [10].

From Definition 2.1 we easy obtain the following Lemma.

Lemma 2.1. *Let K, U_1 and U_2 be a convex subsets of normed space X and the map $F : K \multimap X$ is convex, then the map $(x,y) \mapsto ||F(x) + U_1|| + ||F(y) + U_2||$ is a convex.*

Definition 2.2. (K. Nikodem, [24]). A map $F : X \multimap X$ is called quasi-convex if

$$F(x_i) \cap S \neq \emptyset,\ i = 1, 2 \Rightarrow F(\lambda x_1 + (1-\lambda)x_2) \cap S \neq \emptyset, \tag{3}$$

for all convex sets $S \subset X$, $x_1,\ x_2 \in C$ and $\lambda \in [0,1]$.

Lemma 2.2. *Let K and U be a convex subsets of normed space X if the map $G : K \multimap X$ is quasi-convex, then the map $x \mapsto ||G(x) + U||$ is a quasi-convex.*

Definition 2.3. (J. Dugundji, A. Granas, [13]) Let K be a nonempty subset of a topological vector space X. A map $H : K \multimap X$ is called KKM map if for every finite set $\{x_1, \dots, x_n\} \subset K$, we have

$$\mathrm{conv}\{x_1, \dots, x_n\} \subseteq \bigcup_{k=1}^{n} H(x_k).$$

We using definition of measure of non-compactness of L. Pasicki [25].

Definition 2.4. (L. Pasicki, [25]) Let X be a metric space. Measure of non-compactness on X is an arbitrary map $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies following conditions:
1) $\phi(A) = 0$ if and only if A is totally bounded set;
2) from $A \subseteq B$ follows $\phi(A) \leq \phi(B)$.
5) for each $A \subseteq X$ and $x \in X$ $\phi(A \cup \{x\}) = \phi(A)$.

In next section the following theorem will be needed.

Theorem 2.1. *(Z. D. Mitrović, I. D. Aranđelović, [23]) Let X be a metric linear space, K a nonempty convex complete subset of X, ϕ measure of non-compactness on X and $f, g, h : K \times K \to \mathbb{R}$ are continuous maps such that*

1. *$f(x, x) \leq g(x, x)$ for all $x \in K$,*
2. *$g(x, \lambda y_1 + (1 - \lambda) y_2) \leq \max\{h(x, y_1), h(x, y_2)\}$ for all $x, y_1, y_2 \in K$ and $\lambda \in [0, 1]$,*
3. *for each $t > 0$ there exists $y \in K$ such that*

$$\phi(\{x \in K : f(x, x) \leq h(x, y)\}) \leq t,$$

then there exists a point $x_0 \in K$ such that

$$h(x_0, y) \geq f(x_0, x_0) \text{ for each } y \in K.$$

3 Results

First, we present the following best approximations theorem.

Theorem 3.1. *Let $(X, ||\cdot||)$ be a normed space, K a nonempty convex complete subset of X, ϕ measure of non-compactness on X and $F : K \multimap X$ and $G : K \multimap K$ continuous maps with compact convex values. If there exists a quasi-convex onto map $H : K \multimap K$ such that*

$$||G(x) - F(x)|| \leq ||H(x) - F(x)|| \text{ for all } x \in K \tag{4}$$

and for all $t > 0$ exists $z \in K$ such that

$$\phi(\{x \in K : ||G(x) - F(x)|| \leq ||H(z) - F(x)||\}) \leq t. \tag{5}$$

Then there exists $x_0 \in K$ *such that*

$$||G(x_0) - F(x_0)|| = \inf_{x \in K} ||H(x) - F(x_0)||. \tag{6}$$

Proof. Define

$$f(x, y) = ||G(y) - F(x)|| \text{ and } g(x, y) = ||H(y) - F(x)|| \text{ for } x, y \in K.$$

Now, the result follows by Theorem 2.1 and Lemma 2.2. □

Remark 3.1. If F, G and H are the single-valued maps from Theorem 3.1 we obtain the result of [22], (Theorem 3. 2).

Further we give rhe following coupled best approximations theorem.

Theorem 3.2. *Let* $(X, || \cdot ||)$ *be a normed space,* K *a nonempty convex complete subset of* X*,* ϕ *measure of non-compactness on* $X \times X$ *and* $F_i : K \times K \multimap X$*,* $G_i : K \multimap K$ *continuous maps with compact convex values* $i = 1, 2$*. If there exists a convex onto map* $H : K \multimap X$ *such that*

$$||G_i(x) - F_i(x, y)|| \leq ||H(x) - F_i(x, y)||, \text{ for all } x, y \in K \text{ and } i = 1, 2, \tag{7}$$

and for all $t > 0$ *exists* $z \in K$ *such that*

$$\phi(\{(x, y) \in K \times K : ||G_i(x) - F_i(x, y)|| \leq ||H(z) - F_i(x, y)||\}) \leq t, i = 1, 2. \tag{8}$$

Then there exists $(x_0, y_0) \in K \times K$ *such that*

$$\begin{aligned} ||G_1(x_0) - F_1(x_0, y_0)|| + ||G_2(y_0) - F_2(y_0, x_0)|| = \\ \inf_{(x,y) \in K \times K} \{||H(x) - F_1(x_0, y_0)|| + ||H(y) - F_2(y_0, x_0)||\}. \end{aligned} \tag{9}$$

Proof. Let $f, g : (K \times K) \times (K \times K) \to \mathbb{R}$ defined by

$$f((x_1, y_1), (x_2, y_2)) = ||G_1(x_2) - F_1(x_1, y_1)|| + ||G_2(y_2) - F_2(y_1, x_1)||$$

$$g((x_1, y_1), (x_2, y_2)) = ||H(x_2) - F_1(x_1, y_1)|| + ||H(y_2) - F_2(y_1, x_1)||.$$

Now, the result follows by Theorem 2.1 and Lemma 2.1. □

Remark 3.2. From Theorem 3.2 we obtain Theorem 2.3. of A. Amini-Harandi [4] and the result of [20], (Theorem 2. 1).

Example 3.1. Let $C = [0, 1]$ and define the maps $F, G, H : C \multimap C$ by

$$F(x) = \{0\}, \ G(x) = [0, x(2x - 1)^2], \ H(x) = [0, x].$$

Then map G is not quasi-convex, note that the maps F, G and H satisfy all hypotheses of Theorem 3.1.

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O invarijantama u strukturama prvog reda

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1 Uvod

Jedan od osnovnih zadataka Teorije modela je klasifikacija struktura prvog reda: za datu teoriju prvog reda odrediti uslove pod kojima postoji sistem invarijanta kojima bi se, do na izomorfizam, opisao svaki model teorije. Najpoznatije klasa struktura koje možemo opisati su vektorski prostori nad fiksiranim poljem: do na izomorfizam su odredjeni jednim kardinalnim brojem, svojom dimenzijom. Algebarski zatvorena polja opisujemo sa dva broja: karakteristikom i stepenom transcendentnosti. Svaka Abelova grupa je opisiva nizom trojki kardinala (Ulmove invarijante). Šelah je u svojoj čuvenoj monografiji Classification Theory rešio problem mogućnosti klasifikacije klase neprebrojivih modela date potpune teorije prvog reda u prebrojivom jeziku. Pokazao je da je u klasama u kojima je klasifikacija moguća, invarijanta svakog modela drvo visine ω čiji su čvorovi označeni kardinalnim brojevima; u ostalim klasama moguće je kodirati stacionarne podskupove neprebrojivih kardinala, čija je struktura komplikovana. Primeri neklasifikabilnih klasa su klasa linearnih uredjenja i uopšte bilo koja aksiomatizabilna klasa struktura prvog reda u kojima je moguće definisati parcijalno uredjenje sa beskonačnim lancima.

Prema Šelahovoj teoriji jednostavne invarijante u strukturama su kardinali koji spajanjem sa strukturom ω-drveta čine složene invarijante. Kardinali se pojavljuju u strukturama kao dimenzije tzv. pravilnih tipova. Pravilni tipovi indukuju na prirodan način kombinatornu strukturu predgeometrije (ekvivalent matroida u konačnom slučaju) u svakom modelu teorije. U svakoj predgeometriji se, slično vektorskim prostorima, na prirodan način definiše pojam nezavisnosti i baze, tako da svake dve baze imaju isti kardinalni broj elemenata. Na taj način dobijamo dobro definisanu dimenziju pravilnog tipa u datoj strukturi. Na primer, dimenzija vektorskog prostora i stepen transcendentnosti algebarski zatvorenog polja su dimenzije odredjenih tipova.

Šelahova teorija ne rešava problem klasifikacije prebrojivih modela date teorije. Na primer klasa neprebrojivih gustih linearnih uredjenja nije klasifikabilna, dok klasa prebrojivih gustih linearnih uredjenja sadrži samo četiri dobro poznata tipa

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izomorfizma, u zavisnosti od toga da li i koje krajnje tačke postoje. To je motivacija zbog koje su u [1] uvedene i asimetrične invarijante: linearni uredjajni tipovi. Pod linearnim uredjajnim tipom podrazumeva se klasa ekvivalencije u odnosu na relaciju izomorfizma linearnih uredjenja. I simetrične i asimetrične invarijante proističu iz izvesnih operatora algebarskog zatvaranja odredjenih pravilnim tipovima, što ćemo i prikazati u ovom radu. Uvodni deo rada sadrži prilično detaljne dokaze, dok u ostalom delu prikazujemo neke od rezultata iz [1] i [2].

2 Operatori algebarskog zatvaranja

U ovom poglavlju bavimo se operatorima algebarskog zatvaranja i predgeometrijama. Naizgled, predgeometrije su čisto kombinatorne strukture u kojima je pojam zavisnosti (kao i nezavisnosti i baze) prirodno definisan; on uopštava pojmove linearne zavisnosti u vektorskim prostorima i algebarske zavisnosti u poljima. U narednom poglavlju dokazaćemo da svake dve baze (jedne te iste) predgeometrije imaju isti kardinalni broj elemenata, taj broj je dimenzija predgeometrije. Predgeometrija može biti i beskonačno-dimenziona, kao što je vektorski prostor realnih brojeva nad poljem racionalnih brojeva. Glavni rezultat ovog poglavlja opisuje kako se linearni uredjajni tipovi javljaju kao invarijante u jednoj klasi pravih operatora algebarskog zatvaranja.

Definicija 2.1. Neka je V skup i $\mathrm{cl} : \mathcal{P}(V) \longrightarrow \mathcal{P}(V)$ preslikavanje (odnosno operator na skupu V). cl je operator algebarskog zatvaranja ako su zadovoljene sledeće tri aksiome:

P1	Monotonost	$X \subseteq Y$ povlači $X \subseteq \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;
P2	Idempotentnost	$\mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$;
P3	Konačan karakter	$\mathrm{cl}(X) = \bigcup\{\mathrm{cl}(X_0) \mid X_0 \subseteq X$ je konačan$\}$;

(V, cl) je predgeometrija ukoliko je zadovoljena i aksioma

P4	Aksioma zamene	$x \in \mathrm{cl}(X, y) - \mathrm{cl}(X)$ povlači $y \in \mathrm{cl}(X, x)$.

Pravi operator algebarskog zatvorenja je onaj koji ne koji ne zadovoljava aksiomu zamena.

cl je *operator zatvaranja* ukoliko zadovoljava P1-P2. Primer takvog oeratora je operator zatvaranja u topološkom prostoru. Ispunjenost aksiome P3 isključuje beskonačne Hausdorfove topološke prostore, pa se takvi operatori prirodno javljaju u algebarskim, i šire, u strukturama prvog reda. $\mathrm{cl}(A)$ je *zatvarač* skupa $A \subseteq V$, a skupovi oblika $\mathrm{cl}(A)$ su *zatvoreni* skupovi. $a \in \mathrm{cl}(A)$ čitamo i kao *a zavisi od A* dok $a \notin \mathrm{cl}(A)$ čitamo i kao *a ne zavisi od A*. Ova terminologija potiče iz vektorskih prostora i polja.

Primer 2.1. (1) Linearna zavisnost. Neka je $(V, +, -, \cdot_f, 0)_{f \in F}$ bektorki prostor nad poljem F. Za $X \subseteq V$ definišemo $\mathrm{cl}(X)$ kao potprostor generisan sa X. Nije teško proveriti da je (V, cl) predgeometrija.

(2) Algebarska zavisnost. Neka je $(K, +, -, \cdot, 0, 1)$ polje i F_0 njegovo osnovno polje ($\mathbb{Q}$ ili $\mathbb{Z}_p$). Za $X \subseteq K$ definišimo $\mathrm{acl}(X)$ kao skup svih elemenata polja K koji su algebarski nad poljem $F_0(X)$. Tada je (K, acl) predgeometrija.

Primer 2.2. Neka je $(A, <)$ drvo: $<$ je netrivijalno striktno uredjenje i za svaki $a \in A$ interval $(-, a] = \{x \in A \,|\, x \leqslant a\}$ je linearno uredjen. U specijalnom slučaju $(A, <)$ može biti i linearno uredjenje. Posmatrajmo operator cl definisan sa $\mathrm{cl}(X) = \bigcup_{x \in X}(-, x]$. Nije teško proveriti da je cl pravi operator algebarskog zatvorenja na A.

Definišimo pojam lokalizacije operatora algebarskog zatvarenja cl na skupu V: ako je $A \subseteq V$ tada i $\mathrm{cl}^A(X) = \mathrm{cl}(X \cup A)$ definiše operator algebarskog zatvaranja na skupu V.

Lema 2.1. 1. $\mathrm{cl}(A) = \mathrm{cl}(B)$ ako i samo ako $A \subseteq \mathrm{cl}(B)$ i $B \subseteq \mathrm{cl}(A)$.

2. Presek zatvorenih skupova je zatvoren. □

Definicija 2.2. Neka je cl operator zatvaranja na skupu V i $\mathbb{I} = (I, <)$ linearno uredjenje. Za niz $(a_i \,|\, i \in I)$ elemenata skupa V kažemo da je cl-slobodan ako za sve $i \in I$ važi $a_i \notin \mathrm{cl}(a_j \,|\, j < i)$.

Nije teško uvideti da cl-slobodni nizovi postoje u svakoj strukturi u kojoj je $\mathrm{cl}(\emptyset) \neq V$. Ordinalne cl-slobodne nizove konstruišemo induktivno sledećom procedurom. U prvom koraku uzmimo $a_0 \notin \mathrm{cl}(\emptyset)$. Pretpostavimo da je niz $(a_i \,|\, i < \xi)$ konstruisan; ukoliko je njegov zatvarač ceo skup V konstrukcija se završava, a u suprotnom biramo $a_\xi \in V - \mathrm{cl}(a_i \,|\, i < \xi)$. Ovaj proces se mora završiti na nekom ordinalnom koraku α, t.j kada uslov $\mathrm{cl}(a_i \,|\, i < \alpha) = V$ bude ispunjen. Takve nizove zovemo maksimalnim ordinalnim nizovima. U opštem slučaju ordinal α nije jedinstveno odredjen i za razne izbore elemenata niza možemo dobiti različite, čak i po kardinalnosti, ordinale. To pokazuje sledeći primer.

Primer 2.3. Posmatrajmo operator cl iz dela (1) definisan na $(\omega + 1, <)$. Nizovi $(a_\xi = \xi \,|\, \xi \in \omega + 1)$ i $(a_0 = \omega + 1)$ su maksimalni ordinalni nizovi čije su dužine različitih kardinalnosti.

Definicija 2.3. Operator algebarskog zatvaranja cl na skupu A je totalno degenerisan ako za svaki konačan $X \subseteq A$ postoji $x \in X$ takav da je $\mathrm{cl}(x) = \mathrm{cl}(X)$.

Nije teško utvrditi da je totalna degenerisanost ekvivalentna uslovu: za sve x, y važi $\mathrm{cl}(x, y) = \mathrm{cl}(x)$ ili $\mathrm{cl}(x, y) = \mathrm{cl}(y)$. Ova činjenica je ključna u dokazu naredne leme.

Lema 2.2. Neka je cl totalno degenerisan operator algebarskog zatvaranja na skupu A. Za $x \in A$ neka je $\epsilon_{cl}(x) = \{y \in A \,|\, \mathrm{cl}(x) = \mathrm{cl}(y)$.

1. Skup $\{\mathrm{cl}(x) \,|\, x \in A\}$ je linearno uredjen inkluzijom.

2. Skup $A_{cl} = \{\epsilon_{cl}(x) \,|\, x \in A\}$ je linearno uredjen relacijom

$$\epsilon_{cl}(x) <_{cl} \epsilon_{cl}(y) \text{ ako i samo ako } \mathrm{cl}(x) \subsetneq \mathrm{cl}(y).$$

3. Niz $(a_i \,|\, i \in I\}$ je cl-slobodan ako i samo ako je $(\epsilon_{cl}(a_i) \,|\, i \in I)$ strogo $<_{cl}$-rastući.

Naredna teorema opisuje poreklo uredjajnih tipova kao invarijanata.

Teorema 2.1. Ako je cl totalno degenerisani operator na skupu A, tada za svaki $B \subseteq A$ važi: svaka dva maksimalna cl-slobodna niza elemenata skupa B imaju isti uredjajni tip.

3 Predgeometrije

Naizgled, predgeometrije su čisto kombinatorne strukture u kojima je pojam zavisnosti (kao i nezavisnosti i baze) prirodno definisan; on uopštava pojmove linearne i algebarske zavisnosti. Glavni rezultat ovog poglavlja tvrdi da svake dve baze (jedne te iste) predgeometrije imaju isti kardinalni broj elemenata, taj broj je dimenzija predgeometrije. Predgeometrija može biti i beskonačno-dimenziona, kao što je vektorski prostor realnih brojeva nad poljem racionalnih brojeva.

Do kraja poglavlja, fiksirajmo predgeometriju (V, cl). Za neprazan skup $A \subseteq V$ kažemo da je *nezavisan* ukoliko za sve $a \in A$ važi $a \notin \mathrm{cl}(A - a)$; drugim rečima a ne zavisi od ostalih elemenata skupa A. Maksimalan u odnosu na inkluziju nezavisan skup se naziva *baza.* Aksioma izbora nam garantuje da se svaki nezavisan skup može proširiti do baze. Cilj ovog dela je Teorema o jednakobrojnosti baza.

Lema 3.1. 1. A je baza ako i samo ako je A nezavisan i $\mathrm{cl}(A) = V$.

2. Ako $a \in \mathrm{cl}(C, b) - \mathrm{cl}(C)$ tada $\mathrm{cl}(C, b) = \mathrm{cl}(C, a)$.

U predgeometrijama pojmovi cl-slobodnog niza i nezavisnog skupa su identični. Primetimo da, kako god poredjali elemente nezavisnog skupa u niz, uvek dobijemo cl-slobodan niz. Obrnuti smer je sadržan u sledećoj lemi

Lema 3.2. Elementi cl-slobodnog niza čine nezavisan skup.

Dokaz. Zbog konačnog karaktera dovoljno je dokazati tvrdjenje za konačne nizove. Dokazujemo ga indukcijom po dužini niza. Pretpostavimo da tvrdjenje važi za nizove dužine n. Neka je $A = (a_1, \ldots, a_{n+1})$ cl-slobodan niz. Ukoliko nije nezavisan kao skup, postoji njegov element a_j koji pripada zatvorenju preostalih elemenata. Kako je niz A cl-slobodan važi $a_{n+1} \notin \mathrm{cl}((a_1, \ldots, a_n)$, pa važi $j < n + 1$. Neka je $A' = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)$. Dakle $a_j \in \mathrm{cl}(A', a_{n+1})$. Zbog nezavisnosti skupa $(a_1, \ldots, a_n)$ važi $a_j \notin \mathrm{cl}(A')$ pa možemo primeniti aksiomu zamene: $a_{n+1} \in \mathrm{cl}(A', a_j) = \mathrm{cl}(a_1, \ldots, a_n)$. Kontradikcija. □

Stav 3.1. Ako su A i B nezavisni skupovi i $B \subseteq \mathrm{cl}(A)$, tada važi $|B| \leq |A|$.

Dokaz. Razmotrimo prvo slučaj kada je A konačan skup. Neka su $b_1, ..., b_m$ različiti elementi skupa B. Tvrdimo da postoje medjusobno različiti elementi $a_1, ..., a_m$ skupa A takvi da za svako $k \leq m$ važe sledeći uslovi:

- $A_k = \{b_1, \dots, b_k\} \cup (A - \{a_1, \dots, a_k\})$ je nezavisan skup; $A_0 = A$.
- $\mathrm{cl}(A_k) = \mathrm{cl}(A)$.

Primetimo da ovo tvrdjenje povlači $|B| \leqslant |A|$.

Konstrukciju skupova A_k vršimo indukcijom po $k \leqslant m$. Pretpostavimo da je A_k konstruisan za neko $k < m$. Poredjajmo njegove elemente u niz $(b_1, \dots, b_k, a'_1, \dots, a'_n)$. Kako $b_{k+1} \in \mathrm{cl}(A)$ i prema induktivnoj hipotezi važi $\mathrm{cl}(A_k) = \mathrm{cl}(A)$, zaključujemo da $b_{k+1} \in \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_n)$. Zbog nezavisnosti skupa B važi $b_{k+1} \notin \mathrm{cl}(b_1, ..., b_k)$ pa postoji $i \leqslant n$ takav da

$$b_{k+1} \in \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_i) - \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}). \tag{1}$$

Dokazaćemo da $a_{k+1} = a'_i$ zadovoljava željene uslove. Prvo dokazujemo:

$$(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n, b_{k+1}) \text{ je nezavisan niz.} \tag{2}$$

Nezavisnost ovog niza sa izostavljenim poslednjim članom je posledica nezavisnosti skupa A_k, pa preostaje da dokažemo da $b_{k+1} \notin \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n)$. Pretpostavimo suprotno:

$$b_{k+1} \in \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n). \tag{3}$$

Primenom aksiome zamene na (1) dobijamo:

$$a'_i \in \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}, a'_{i+1}, b_{k+1}) \tag{4}$$

pa, koristeći (3) i idempotentnost, zaključujemo

$$a'_i \in \mathrm{cl}(b_1, \dots, b_k, a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n). \tag{5}$$

To je u kontradikciji sa induktivnom hipotezom da je skup A_k nezavisan. Time smo dokazali (2). Prema Lemi 3.2 skup A_{k+1} je nezavisan, čime smo dokazali ispunjenost prvog uslova. Preostaje da proverimo drugi uslov: $\mathrm{cl}(A_{k+1}) = \mathrm{cl}(A)$. Imamo:

$$\mathrm{cl}(A_{k+1}) = \mathrm{cl}(\mathrm{cl}(A_{k+1})) \supseteq \mathrm{cl}(A_{k+1}, a'_i) \supseteq \mathrm{cl}(A_k)\,,$$

gde jednakost važi zbog idempotentnosti, prva inkluzija zbog (4) i monotonosti, a druga inkluzija zbog monotonosti. Time smo dokazali $\mathrm{cl}(A_{k+1}) \supseteq \mathrm{cl}(A_k) = \mathrm{cl}(A)$ pa, zbog $A_{k+1} \subseteq A$ i monotonosti imamo $\mathrm{cl}(A_{k+1}) = \mathrm{cl}(A)$. Time je dokaz u slučaju kada je A konačan skup završen.

Pretpostavimo sada da je $|A| = \kappa$ beskonačan. Postoji κ različitih konačnih podskupova skupa A, pa postoji najviše κ različitih skupova oblika $\mathrm{cl}(A_0)$ gde je

A_0 konačan podskup od A. Fiksirajmo za momenat konačan skup $A_0 \subset A$. Neka je $B_0 = B \cap \mathrm{cl}(A_0)$. Tada su A_0 i B_0 nezavisni skupovi i važi $\mathrm{cl}(B_0) \subseteq \mathrm{cl}(A_0)$. Kako je A_0 konačan skup možemo primeniti dokazani deo teoreme i zaključiti $|B_0| \leq |A_0|$. Posebno, $B_0 = B \cap \mathrm{cl}(A_0)$ je konačan. Time smo dokazali da svaki $\mathrm{cl}(A_0)$ sadrži konačno mnogo elemenata skupa B. S druge strane, svaki element skupa B se nalazi u nekom od skupova oblika $\mathrm{cl}(A_0)$ prema konačnom karakteru predgeometrije. Prema tome, imamo κ skupova $\mathrm{cl}(A_0)$ i svaki od njih sadrži konačno mnogo elemenata skupa B. Prema tome $|B| \leq \kappa$. □

Posledica 3.1. Ako su A i B nezavisni skupovi i $\mathrm{cl}(A) = \mathrm{cl}(B)$ tada je $|A| = |B|$.

Teorema 3.1. Svake dve baze predgeometrije imaju jednake kardinalne brojeve.

Broj elemenata baze se naziva *dimenzija* predgeometrije. Definišemo dimenziju svakog podskupa: ako je $A \subseteq V$ tada svaka dva njegova nezavisna podskupa imaju istu kardinalnost, koja se zove dimenzija skupa A u predgeometriji.

4 Pravilni tipovi

Neka je $(M, \ldots)$ struktura jezika $\mathcal{L}$ i $A \subseteq M$. Tip po promenljivoj x nad $A \subset M$ je bilo koji skup $\mathcal{L}_A$-formula (sa parametrima iz A) koji nema drugih slobodnih promenljivih osim x, takav da je svaki njegov konačan podskup zadovoljiv (a samim tim i realizovan u M). Tip je potpun ako za svaku takvu formulu važi da ili ona ili njena negacija pripadaju tipu. Podsetimo da je model κ-zasićen ako realizuje sve tipove sa $< \kappa$ parametara iz njegovog domena. U κ-zasićen model elementarno se može utopiti svaka elementarno ekvivalentna struktura moći $< \kappa$. Model M je κ-homogen (u jakom smislu) ako se svako parcijalno elementarno preslikavanje skupa $A \subset M$ ($|A| < \kappa$) u M može produžiti do automorfizma modela M; $f : A \longrightarrow M$ je parcijalno elementarno preslikavanje ako za sve $\bar{a} \in A$ i svaku formula $\phi(\bar{x})$ važi: $M \models \phi(\bar{a})$ ako i samo ako $M \models \phi(f(\bar{a}))$.

Neka je T fiksirana, kompletna teorija prebrojivog jezika $\mathcal{L}$ koja ima beskonačne modele. Zanimaju nas prebrojivi modeli teorije T. Fiksirajmo $\aleph_1$-zasićen, $\aleph_1$-homogen model $\mathbb{U}$ teorije T (takozvani monstrum-model ili univerzum). Zbog zasićenosti svaki prebrojiv model teorije T je izomorfan nekom elementarnom podmodelu univerzuma. Zato ćemo raditi isključivo unutar univerzuma. Sa $a, b, ...$ označavamo elemente, sa $A, B, ...$ njegove najviše prebrojive podskupove, a sa $M, M_1, ...$ označavamo domene elementarnih podmodela univerzuma.

Globalni tip je potpun 1-tip nad $\mathbb{U}$, takve tipove označavamo sa $\mathfrak{p}, \mathfrak{q}, \ldots$ Za svaki A definišemo definšemo restrikciju $\mathfrak{p} \restriction A = \{\phi(x) \in \mathfrak{p} \mid \phi(x) \text{ je } \mathcal{L}_A\text{-formula}\}$. $\mathfrak{p}$ je A-invarijantan ako za svaku A-formulu $\phi(x; \bar{y})$ i za sve n-torke $\bar{b}_1, \bar{b}_2$ važi:

$$\mathrm{tp}(\bar{b}_1/A) = \mathrm{tp}(\bar{b}_2/A) \text{ povlači } (\phi(x, \bar{b}_1) \Leftrightarrow \phi(x, \bar{b}_2)) \in \mathfrak{p}.$$

Neka je $\mathfrak{p}$ A-invarijantan tip. Niz $(a_i \mid i < \alpha)$ je Morlijev niz tipa $\mathfrak{p}$ nad A ako a_β realizuje $\mathfrak{p} \restriction A \cup \{a_\xi \mid \xi < \beta\}$ za svaki $\beta < \alpha$. Za regularne tipove Morlijevi nizovi su isto što i cl-slobodni nizovi: ako je $\mathfrak{p}$ regularan tada je niz $(a_i \mid i < \alpha)$ Morlijev niz tipa $\mathfrak{p}$ nad A ako i samo ako je $\mathrm{cl}_{\mathfrak{p}}^A$-slobodan.

Sledeći primer ukazuje na vezu izmedju globalnih tipova i pojma dimenzije

Primer 4.1. Neka je K beskonačno (prebrojivo) polje. Posmatrajmo $\aleph_1$-dimenzioni vektorski prostor $\mathbb{V} = (V, +, k\cdot, 0)_{k \in K}$. Poznato je da ova teorija dopušta eliminaciju kvantifikatora, pa je svaki definabilan (sa parametrima) skup $D \subseteq V$ ili konačan ili ko-konačan (komplement mu je konačan). Označimo sa $\mathfrak{p}$ skup svih formula sa parametrima koje definišu ko-konačan podskup. Tada je $\mathfrak{p}$ potpun $\emptyset$-invarijantan tip. Formule koje su sadržane u $\mathfrak{p}$ definišu 'velike' podskupove, dok njihove negacije definišu 'male' podskupove. Za svaki $A \subset V$ označimo sa $\mathrm{cl}(A)$ uniju svih malih skupova koji su definabilni formulom sa parametrima iz A. Koristeći eliminaciju kvantifikatora nije teško zaključiti da je $\mathrm{cl}(A)$ potprostor generisan sa A. Prema tome, linearna zavisnost u vektorskim prostorima je indukovana operatorom koji je odredjen tipom $\mathfrak{p}$: $\mathrm{cl}(A) = \{\phi(V) \mid \neg\phi(x) \in \mathfrak{p}\restriction A\}$.

Definicija 4.1. Neka je $\mathfrak{p}$ globalan tip. Definišemo operator $\mathrm{cl}_{\mathfrak{p}}$ na $\mathbb{U}$ na sledeći način: $\mathrm{cl}_{\mathfrak{p}}(X) = \{\phi(V) \mid \neg\phi(x) \in \mathfrak{p}\restriction A\}$ (za $X \subseteq \mathbb{U}$).

U opštem slučaju $\mathrm{cl}_{\mathfrak{p}}$ nije operator algebarskog zatvaranja. U narednoj definiciji, radi jednostavnosti, koristimo pojam 'pravilan tip' u smislu '$(\mathfrak{p}, x = x)$ is strongly regular over A' definisanom u [2] i [1].

Definicija 4.2. Globalni tip $\mathfrak{p}$ je pravilan ako je $\emptyset$-invarijantan, a $\mathrm{cl}_{\mathfrak{p}}$ operator algebarskog zatvaranja na univerzumu.

Stav 4.1. Neka je $\mathfrak{p}$ pravilan globalni tip. Tada je $(\mathbb{U}, \mathrm{cl}_{\mathfrak{p}})$ predgeometrija ako i samo ako je svaki (ekvivalentno bar jedan beskonačan) Morlijev niz simetričan.

Prethodno tvrdjenje nam daje dihotomiju: postoje simetrični i asimetrični pravilni tipovi. Asimetrični su oni za koje postoji nesimetričan konačan Morlijev niz, dok su simetrični oni kojima su svi Morlijevi nizovi simetrični. Simetrični odredjuju kardinalne invarijante: $\dim_{\mathfrak{p}}(M)$ je dimenzija skupa M u predgeometriji indukovanoj operatorom $\mathrm{cl}_{\mathfrak{p}}$.

Teorema 4.1. ([2]) Neka je $\mathfrak{p}$ globalni tip. Ako je $(\mathbb{U}, \mathrm{cl}_{\mathfrak{p}})$ predgeometrija, tada je $\mathfrak{p}$ $\emptyset$-invarijantan, definabilan i pravilan.

Dajemo primer asimetričnog pravilnog tipa i linearnog uredjajnog tipe kao invarijante koju on odredjuje.

Primer 4.2. Neka je univerzum $(\omega + \mathbf{L} \times \mathbb{Z}, <)$; to je linearno uredjenje dobijeno dodavanjem prirodnim brojevima na desni kraj kopije leksikografskog proizvoda nekog $\aleph_1$-zasićenog, gustog linearnog uredjenja i celih brojeva. Poznato je da ako u jezik dodamo i simbol za funkciju neposrednog sledbenika, imamo eliminaciju kvantora. Odatle sledi da je svaki prebrojiv elementarni podmodel M opisan do na izomorfizam uredjajnim tipom svoje projekcije na $\mathbb{L}$; pri tome, svaki podskup od $\mathbf{L}$ je projekcija nekog modela.

Tip izomorfizma projekcije možemo opisati modelsko teoretski: postoji jedistven globalni tip beskonačno velikog elementa; odredjen je skupom formula $\{a < x \,|\, a \in \mathbb{U}\}$. Ispostavi se da je $\mathfrak{p}$ regularan, kao i da skup $\mathrm{cl}_{\mathfrak{p}}(A)$ sačinjen od svih elemenata $b \in \mathbb{U}$ za koje postoje $a \in A$ i $n \in \mathbb{N}$ takvi da važi $b < s^n(a)$. Morlijev niz tipa $\mathfrak{p}$ nad $\emptyset$ je bilo koji niz elemenata skupa $\mathbf{L} \times \mathbb{Z}$ čije prve koordinate čine strogo rastući niz u $\mathbb{L}$. Morlijev niz je maksimalan ukoliko se projektuje bijektivno na projekciju modela M. Prema tome, za svaki M uredjajni tip bilo kog maksimalnog Morlijevog niza je ujedno i uredjajni tip projekcije, pa uredjajni tip maksimalnog Morlijevog niza ne zavisi od izbora samog niza. Svaki prebrojiv elementarni podmodel M je opisan do na izomorfizam uredjajnim tipom svojih maksimalnih Morlijevih nizova.

Naredna teorema utvrdjuje da je u slučaju ma kog asimetričnog tipa situacija slična kao u prethodnom primeru, i da asimetrični tipovi odredjuju uredjajne tipove kao invarijante modela.

Teorema 4.2. [1] Pretpostavimo da je $\mathfrak{p}$ asimetričan pravilan tip, da je A konačan skup i da za svaki (ekvivalentno neki) Morlijev niz a, b tipa $\mathfrak{p}$ nad A važi $\mathrm{tp}(a, b/A) \neq \mathrm{tp}(b, a)$.

(1) Postoji A-definabilno parcijalno uredjenje takvo da je svaki Morlijev niz nad A strogo rastući.

(2) Operator $\mathrm{cl}_{\mathfrak{p}}^{A}$ je totalno degenerisan na $\mathbb{U}$.

(3) Ako $M \supseteq A$ tada svaka dva maksimalna Morlijeva niza sadržana u M imaju isti tip uredjenja. koji se označava sa $\mathrm{Inv}_{\mathfrak{p},A}(M)$.

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Some Properties of Semicommutative Rings

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Original scientific paper

Abstract

Throughout this paper R denotes an associative ring with identity. Recall that a ring is reduced if it has no nonzero nilpotent elements. We used the term reversible to denote zero commutative ring. A generalization of a reversible ring is a semicomutative ring. A ring R is semicommutative if$ab = 0$ implies $aRb = 0$ for $a, b \in R$. We investigate relation between Armendariz and semicommutative rings. We also aproach posibility of transferring semicommutativity property from the ring to some of his extensions.

1 Preliminaries

Throughout this paper R denotes an associative ring with identity. We say that a ring R is semicommutative if For any $a, b \in R$, $ab = 0$ implies $aRb = 0$. A ring R is called weakly semicommutative if for any $a, b \in R$, $ab = 0$ implies $arb \in nil(R)$ for any $r \in R$. Clearly any semicommutative ring is weakly semicommutative. From [3] we know that the convese is not true. We also know that a class of weakly semicommutative rings is closed for some matrix ring extensions such as n-by-n upper matrix ring $T_n(R)$. In the next σ denotes an endomorphism of R and $R[x; \sigma]$ denotes skew polynomial ring with the ordinary addition and the multiplication subject to the relation $xr = \sigma(r)x$. When σ is an automorphism, $R[x, x^{-1}; \sigma]$ denotes skew Laurent polynomial ring with the multiplication subject to the relation $x^{-1}r = \sigma^{-1}(r)x$. The class of weakly semicommutative rings is closed for Laurent formal sum constructions,ie. A ring $R[x]$ is weakly semicommutative if and only if $R[x, x^{-1}]$ is weakly semicommutative. A class of semicommutative rings is not closed under polynomial extensions.C. Huh, Y. Lee and A. Smoktunowicz gave an example of a semicommutative ring R such that $R[x]$ is not semicommutative. ring. Racall that notion of Armendariz ring is introduced by Rege and Chhawchharia [1]. They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all polynomials $f(x) = \sum\limits_{i=0}^{n} a_i x^i$ and $g(x) = \sum\limits_{j=0}^{m} b_j x^j$ from $R[x]$. The motivation for those rings comes from the fact that Armendariz had shown that reduced rings ($a^2 = 0$ implies $a = 0$) satisfy this condition. The notion of Armendariz ring is natural and useful in understanding of the relation between annihilators of rings R and $R[x]$ (see [6]). Those rings were also studied by Armendariz himself, Hong and Kim [7], Chen and Tong [5], Krempa [8] and others.

2 Matrix and polynomial ring extensions

Let R be reduced ring. We define a ring

$$R_n = \left\{ \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ 0 & a_0 & a_1 & \dots & a_{n-2} \\ 0 & 0 & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_0 \end{bmatrix} \mid a_i \in R, i = 0, 1, ..., n-1 \right\}.$$

R_n is not semicommutative for $n \geqslant 2$. Also ring R_n is weakly semicommuative. ([2])This statement does not hold for full matrix ring $M_n(R)$.

For a ring R consider a following set of triangular matrices

$$T_n(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \mid a_{ij} \in R \right\}.$$

It is well known that $T_n(R)$ is subring of the triangular matrix rings with matrix addition and multiplication. Each endomorphism α of a ring R can be naturally extended to a endomorphism

$$\overline{\alpha} : T_n(R) \to T_n(R)$$

with:

$$\overline{\alpha} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix} \right) =$$

$$\begin{bmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \alpha(a_{13}) & \dots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \alpha(a_{23}) & \dots & \alpha(a_{2n}) \\ 0 & 0 & \alpha(a_{33}) & \dots & \alpha(a_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(a_{nn}) \end{bmatrix}$$

Let $E_{ij} = (e_{st} : 1 \leqslant s, t \leqslant n)$ denotes $n \times n$ unit matrices over ring R, in which $e_{ij} = 1$ and $e_{st} = 0$ when $s \neq i$ or $t \neq j$, $0 \leqslant i, j \leqslant n$, for all $n \geqslant 2$. If $V = \sum_{i=1}^{n-1} E_{i,i+1}$, then $V_n(R) = RI_n + RV + ... + RV^{n-1}$ is the subring of upper triangular skew matrices.

Corollary 2.1. Suppose that α is an endomorphism of ring R. If the factor ring $R[x]/(x^n)$ is weak $\widetilde{\alpha}$-skew Armendariz, then $T_n(R)$ is weak $\widetilde{\alpha}$-skew Armendariz.

Proof. Suppose that $R[x]/(x^n)$ is weak $\widetilde{\alpha}-$skew Armendariz and define the ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ by

$$\theta(r_0 I_n + r_1 V + .. + r_{n-1} V^{n-1}) = r_0 + r_1 x + .. + r_{n-1} x^{n-1} + (x^n).$$

Now we have that $V_n(R)$ is weak $\theta^{-1}\widetilde{\alpha}\theta$-skew Armendariz and
$\theta^{-1}\widetilde{\alpha}\theta(r_0 I_n + r_1 V + .. + r_{n-1} V^{n-1}) = \theta^{-1}\widetilde{\alpha}(r_0 + r_1 x + ... + r_{n-1} x^{n-1} + (x^n))$
□

We end this part by transfering property of semicommutativity from ring R to extension $V_n(R)$.

Theorem 2.1. A ring R is weakly semicommutative if and only if ring $V_n(R)$ is weakly semicommutative.

Proof. We define the ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ by

$$\theta(r_0 I_n + r_1 V + .. + r_{n-1} V^{n-1}) = r_0 + r_1 x + .. + r_{n-1} x^{n-1} + (x^n).$$

Now we obtan result from [3] □

In this section we introduce Laurent σ-Armendariz rings and Laurent σ-skew power series rings and we give their useful characterization in terms of σ-skew Armendariz rings. Throughout this section σ is a ring automorphism.

A ring R is σ-skew Armendariz ring of Laurent type if for every two polynomials

$$f(x) = \sum_{i=-p}^{q} a_i x^i, \quad g(x) = \sum_{j=-t}^{s} b_j x^j$$

from $R\left[x, x^{-1}; \sigma\right]$,

$$f(x)g(x) = 0 \ \text{ implies } \ a_i \sigma^i(b_j) = 0, -p \leqslant i \leqslant q, -t \leqslant j \leqslant s.$$

We say that R is σ-skew power series Armendariz ring of Laurent type if for every

$$f(x) = \sum_{i=-p}^{\infty} a_i x^i, \quad g(x) = \sum_{j=-t}^{\infty} b_j x^j$$

from the power series ring $R[[x, x^{-1}; \sigma]]$,

$$f(x)g(x) = 0 \ \text{ implies } \ a_i \sigma^i(b_j) = 0, -p \leqslant i \leqslant \infty, -t \leqslant j \leqslant \infty.$$

In the following two theorems we give a useful characterization of Laurent σ-skew Armendariz rings and Laurent σ-skew power series rings.

Theorem 2.2. The following conditions are equivalent:

1. R is σ-skew Armendariz ring,

2. R is σ-skew Armendariz ring of Laurent type.

Proof. Suppose that $f(x) = \sum_{i=-p}^{q} a_i x^i$ and $g(x) = \sum_{j=-t}^{s} b_j x^j$ are polynomials from the ring $R[x, x^{-1}; \sigma]$ such that $f(x)g(x) = 0$. Since $x^p f(x)$ and $x^t g(x)$ are polynomials from the ring $R[x; \sigma]$ we have that $x^p f(x)g(x)x^t = 0$ which gives $\sigma^p(a_i)\sigma^{i+p}(b_j) = 0, -p \leqslant i \leqslant q, -t \leqslant j \leqslant s$. Since σ is an automorphism,

$$\sigma^p(a_i\sigma^i(b_j)) = 0,$$

so that we have $a_i\sigma^i(b_j) = 0$. The converse is evident since $R[x; \sigma] \subset R\left[x, x^{-1}; \sigma\right]$. □

Theorem 2.3. The following conditions are equivalent:

1. R is σ-skew power series Armendariz ring,

2. R is σ-skew power series Armendariz ring of Laurent type.

Proof. The same as the proof of the previous theorem. □

We close this section with an interesting remark which gives a sufficient condition for the power series ring $R[[x; \sigma]]$ to be reduced.

Theorem 2.4. If an endomorphism σ of a reduced ring R satisfies so called compatibility condition: $a\sigma(b) = 0 \Leftrightarrow ab = 0$, then the power series ring $R[[x; \sigma]]$ is reduced.

Proof. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $(f(x))^2 = 0$. We have to prove that $f(x) = 0$. It is clear that $a_0^2 = 0$, so that $a_0 = 0$. Now, since the coefficient of x^2 has to be zero, we have

$$a_0a_2 + a_1\sigma(a_1) + a_2\sigma^2(a_0) = 0,$$

so that we obtain $a_1\sigma(a_1) = 0$. From the compatibility condition we obtain $a_1^2 = 0$ and since R is reduced, we have $a_1 = 0$. Continuing this way, since the coefficient of x^{2n} is zero, we have $a_n\sigma^n(a_n) = 0$ and, using compatibility condition once again, we have $a_n\sigma^{n-1}(a_n) = 0$ and in the same way $a_n\sigma(a_n) = 0$, so that $a_n = 0$. By induction, we have $a_i = 0$, for all i. This means that $f(x) = 0$ and so the ring $R[[x; \sigma]]$ is reduced. □

Without compatibility condition the previous theorem is not true. Since if the ring $R = Z_2 \bigoplus Z_2$ and σ is defined by $\sigma(a, b) = (b, a)$, it is easy to check that $R[[x; \sigma]]$ is not reduced. Observe that $(1, 0)(0, 1) = (0, 0)$ but $(1, 0)\sigma(0, 1) \neq (0, 0)$.

Recall that a ring R is weak σ-rigid if $a\sigma(a) \in nil(R) \Leftrightarrow a \in nil(R)$. It is easy to see that the notion of weak σ-rigid ring generalizes the notion of a σ-rigid ring. Every homomorphism σ of rings R and S can be extended to the homomorphism of rings $R[x]$ and $S[x]$ by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \sigma(a_i)x^i$, which we also denote by σ. Chen and Tong in [5] prove that if σ is ring isomorphism of rings R and S and R is α-skew Armendariz, then S is $\sigma\alpha\sigma^{-1}$ skew Armendariz ring. We prove the weak skew Armendariz variant of this theorem.

Theorem 2.5. Let R and S be rings with a ring isomorphism $\sigma : R \to S$. If R is weak α-skew Armendariz then S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz.

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ are polynomials from the ring $S[x;\sigma\alpha\sigma^{-1}]$. We have to prove that $f(x)g(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S)$, for all i and j.

As we noted, σ extends to the isomorphism of corresponding polynomial rings, so that there exists polynomials $f_1(x) = \sum_{i=0}^{m} a'_i x^i$ and $g_1(x) = \sum_{j=0}^{m} b'_j x^j$ from $R[x]$ such that $f(x) = \sigma(\ f_1(x)) = \sum_{i=0}^{m} \sigma(a'_i)x^i$ and $g(x) = \sigma(g_1(x)) = \sum_{j=0}^{m} \sigma(b'_j)x^j$.

First, we shall show that $f(x)g(x) = 0$ implies $f_1(x)g_1(x) = 0$. If $f(x)g(x) = 0$, we have

$$a_0 b_k + a_1(\sigma\alpha\sigma^{-1})(b_{k-1}) + ... + a_k(\sigma\alpha\sigma^{-1})^k(b_0) = 0,$$

for any $0 \leqslant k \leqslant m$. From the definition of $f_1(x)$ and $g_1(x)$, we have,

$$\sigma(a'_0)\sigma(b'_k) + \sigma(a'_1)(\sigma\alpha\sigma^{-1})\sigma(b'_{k-1}) + ... + \sigma(a'_k)(\sigma\alpha\sigma^{-1})^k\sigma(b'_0) = 0,$$

so that $(\sigma\alpha\sigma^{-1})^t = \sigma\alpha^t\sigma^{-1}$ we obtain

$$a'_0 b'_k + a'_1\alpha(b'_{k-1}) + ... + a'_k\alpha^k(b'_0) = 0,$$

which means that $f_1(x)g_1(x) = 0$ in the ring $R[x;\alpha]$.

It remains to prove that $f_1(x)g_1(x) = 0$ implies $a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S)$. From the fact that R is weak α-skew Armendariz we have $a'_i\alpha^i(b'_j) \in nil(R)$.

and since $a'_i = \sigma^{-1}(a_i), b'_j = \sigma^{-1}(b_j)$, we have $\sigma^{-1}(a_i)\alpha^i\sigma^{-1}(b_j) \in nil(R)$. This implies

$$\sigma^{-1}(a_i)\sigma^{-1}\sigma\alpha^i\sigma^{-1}(b_j) = \sigma^{-1}(a_i(\sigma\alpha\sigma^{-1})^i(b_j)) \in nil(R)$$

and finally we obtain

$$a_i(\sigma\alpha\sigma^{-1})^i(b_j) \in nil(S),\ 0 \leqslant i, j \leqslant m.$$

Hence S is weak $\sigma\alpha\sigma^{-1}$-skew Armendariz. □

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The boundary value problem with one delay and two potentilas-construction of the solution and asimptotic of eigenvalues

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Abstract

This paper deals with second-order differential operators with one constant delay and two potentials. We consider the boundary value problem $L = L(q_1(x), q_2(x), \tau_2, h, H)$:

$$-y''(x) + q_1(x)y(x) + q_2(x)y(x - \tau_2) = \lambda y(x),\ x \in [0, \pi]$$

$$y(x - \tau_2) \equiv 0,\ x \in [0, \tau_2],$$

$$y(\pi) = 0.$$

By the method of succesive approximation we construct the solution of the differential equation under the initial condition $y(0) = 0$. Then we determine the characteristic function of the operator L and we study asymptotic of eigenvalues.

1 Introduction

Some of the main results for classical Sturm-Liouville operators are presented in [1]-[3], while some of the results for differential operators with delay can be found in papers [4]-[7]. Inverse spectral problems for differential operators with delay have not been studied enough, because some of the main methods in the inverse problem theory for classical Sturm-Liouville operators, such as transformation operator method and method of spectral mappings, are not suitable for differential operators with delay. The class of operators with more than one delay and/or potentials is least studied, but some of the results for this class of operators can be found in [8] and [9]. In this paper we deal with the boundary value problem with one delay and two potentials. In section 2. we construct the solution of the differential equation under the initial condition by the method of successive approximation and determine the characteristic function of the operator L. In Section 3. we study the asymptotic of zeros of the characteristic function in detail. That will be the base for further consideration of the inverse problems for this class of operators by new method based on direct relations between eigenvalues and Fourier's coefficients.

2 Construction of the solution and determining of the characteristic function

We consider the boundary value problem $L = L(q_1(x), q_2(x), \tau_2, h, H)$:

$$-y''(x) + q_1(x)y(x) + q_2(x)y(x-\tau_2) = \lambda y(x),\ \lambda = z^2 \tag{1}$$

$$y(x-\tau_2) \equiv 0,\ x \in [0, \tau_2], \tag{2}$$

$$y(\pi) = 0. \tag{3}$$

where

$$k_0\tau_2 \le \pi < (k_0+1)\tau_2 \tag{4}$$

Firstly, we will determine the integral equation equivalent to the boundary value problem.

Lemma 2.1. *The boundary value problem (1)-(2) for $x \in (\tau_2, \pi]$ is equivalent to the integral equation*

$$\begin{aligned} y(x,z) &= \sin zx + \frac{1}{z}\int_0^x q_1(t_1)\sin z(x-t_1)y(t_1,z)dt_1 + \\ &+ \frac{1}{z}\int_{\tau_2}^x q_2(t_1)\sin z(x-t_1)y(t_1-\tau_2,z)dt_1 \end{aligned} \tag{5}$$

while for $x \in (0, \tau_2]$, it is equivalent to the integral equation

$$y(x,z) = \sin zx + \frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)y(t_1,z)dt_1 \tag{6}$$

Proof 2.1. Using the method of variation of constants, we get the integral equation (5), and then from (5) and (2), we get (6). Let us introduce the following:

$$\begin{aligned}
&b_{s^2}(x,z) = \int\limits_0^x q_1(t_1)\sin z(x-t_1)\sin zt_1 dt_1,\\
&b_{s^{l+1}}(x,z) = \int\limits_0^x q_1(t_1)\sin z(x-t_1)b_{s^l}(t_1,z)dt_1, \;\; l=2,3,...\\
&b_{s^2,\tau_2}(x,z) = \int\limits_{\tau_2}^x q_2(t_1)\sin z(x-t_1)\sin z(t_1-\tau_2)dt_1,\\
&b_{s^{l+1},l\tau_2}(x,z) = \int\limits_{l\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s^l,(l-1)\tau_2}(t_1-\tau_2,z)dt_1, \;\; l=2,3,...\\
&b^{(1,2)}_{s^3,\tau_2}(x,z) = \int\limits_{\tau_2}^x q_1(t_1)\sin z(x-t_1)b_{s^2,\tau_2}(t_1,z)dt_1,\\
&b^{(2,1)}_{s^3,\tau_2}(x,z) = \int\limits_{\tau_2}^x q_2(t_1)\sin z(x-t_1)b_{s^2}(t_1-\tau_2,z)dt_1,
\end{aligned} \tag{7}$$

Let $S_l(l-k,k)$ denote a set of all permutations with repetition with $l-k$ $1's$ and k $2's$, and $S^{(i)}_l(l-k,k)$ denote a subset of $S_l(l-k,k)$ of permutations beginning with i, $i=1,2$. Let us denote

$$b^P_{s^{l+1},k\tau_2}(x,z)|_{P\in S^{(1)}_l(l-k,k)} = \int\limits_{k\tau_2}^x q_1(t_1)\sin z(x-t_1)b^P_{s^l,k\tau_2}(t_1,z)dt_1, \tag{8}$$

$$k=1,2,...k_0,\;\; l=k+1,...$$

where the permutation P in $b^P_{s^{l+1},k\tau_2}(x,z)|_{P\in S^{(1)}_l(l-k,k)}$ from (8) is formed by adding 1 at the beginning of the permutation P in $b^P_{s^l,k\tau_2}(x,z)$, and

$$b^P_{s^l,(l-1)\tau_2}(x,z) = b_{s^l,(l-1)\tau_2}(x,z),\;\; l=2,...,k_0.$$

Let us also denote

$$b^P_{s^{l+1},k\tau_2}(x,z)|_{P\in S^{(2)}_l(l-k,k)} = \int\limits_{k\tau_2}^x q_2(t_1)\sin z(x-t_1)b^P_{s^l,(k-1)\tau_2}(t_1-\tau_2,z)dt_1, \tag{9}$$

$k = 1, 2, ...k_0,\ l = k + 1, ...$ where the permutation P in $b^P_{s^{l+1},k\tau_2}(x,z)|_{P\in S^{(2)}_l(l-k,k)}$ from (9) is formed by adding 2 at the beginning of the permutation P in $b^P_{s^l,(k-1)\tau_2}(x,z)$, and for $k = 1$

$$b^P_{s^{l+1},(k-1)\tau_2}(x,z) = b^P_{s^{l+1}}(x,z) = b_{s^{l+1}}(x,z),\ l = 1, 2, ...$$

Theorem 2.1. *If $q_1, q_2 \in L_2[0,\pi]$ and $k_0\tau_2 \le \pi < (k_0+1)\tau_2$, then the solution of the boundary value problem (1)-(3) within the interval $(k_0\tau_2, \pi]$ has the form:*

$$y(x,z) = \sin zx + \frac{1}{z}b_{s^2}(x,z) + \frac{1}{z}b_{s^2,\tau_2}(x,z) + \sum_{k=2}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(x,z)+$$

$$+\sum_{k=2}^{k_0}\frac{1}{z^k}b_{s^{k+1},k\tau_2}(x,z) + \sum_{i=1}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}b^P_{s^{k+1},i\tau_2}(x,z) \qquad (10)$$

Proof 2.2. We solve integral equations (5) and (6) by the method of successive aproximations, using the following reccurent formula:

$$y^{(i)}_k(x,z) = \frac{1}{z}\int_{i\tau_2}^{x} q_1(t_1)\sin z(x-t_1)y^{(i)}_{k-1}(t_1,z)dt_1+$$

$$+\frac{1}{z}\int_{i\tau_2}^{x} q_2(t_1)\sin z(x-t_1)y^{(i-1)}_k(t_1-\tau_2,z)dt_1,\ x > i\tau_2 \qquad (11)$$

$y^{(i)}_k(x,z) = 0$ for $x \le i\tau_2,\ i = 0, 1, ...k_0,\ k = 0, 1, 2...$
where

$$y^{(0)}_0(x,z) = y_0(x,z) = \sin zx;\ y^{(0)}_k(x,z) = y_k(x,z),\ k = 1, 2, ...$$

$y^{(i-1)}_k(x,z) = 0$ for $i = 0,\ k = 1, 2, ..;$ $y^{(i)}_{k-1}(x,z) = 0$ for $k = 0,\ i = 1, 2, ..k_0$.
In order to simplify, hereinafter we will write the values of the functions from the recurrent formula (11) only for $x > i\tau_2$, assuming that they are equal to zero for $x \le i\tau_2$. From the reccurent formula (11) it is obvious that for $i = 0$ we get the solution of the integral equation within the interval $(0, \tau_2]$, for $i = 0, 1$ we get the solution within the interval $(\tau_2, 2\tau_2]$, and for $i = 0, 1, ...n$ we get the solution of the integral equation within the interval $(n\tau_2, (n+1)\tau_2]$, $n = 2, 3, ...k_0$.

1. Firstly, we will determine the solution within the interval $(0, \tau_2]$. For $i = 0$ from (11) we get

$$y_k(x,z) = \frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)y_{k-1}(t_1,z)dt_1;\ x>0,\ k=1,2,...;\ y_0(x,z)=\sin zx.$$

Then we have

$$y_1(x,z) = \frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)y_0(t_1,z)dt_1$$

$$= \frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)\sin zt_1 dt_1 = \frac{1}{z}b_{s^2}(x,z),$$

and easliy show that

$$y_k(x,z) = \frac{1}{z^k}b_{s^{k+1}}(x,z),\ k=2,3,... \quad (12)$$

From (11) and (12) we get the solution of the boundary value problem within the interval $(0, \tau_2]$ in the form:

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) = \sin zx + \frac{1}{z}b_{s^2}(x,z) + \sum_{k=2}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(x,z) \quad (13)$$

2. Let us now determine functions $y_k^{(1)}(x,z)$, $k = 0,1,2...$
Since from (11) $y_{k-1}^{(1)}(x,z) = 0$ for $k = 0$, we get

$$y_0^{(1)}(x,z) = \frac{1}{z}\int\limits_{\tau_2}^x q_2(t_1)\sin z(x-t_1)y_0(t_1-\tau_2,z))dt_1$$

$$= \frac{1}{z}\int\limits_{\tau_2}^x q_2(t_1)\sin z(x-t_1)\sin z(t_1-\tau_2)dt_1 = \frac{1}{z}b_{s^2,\tau_2}(x,z).$$

Further,

$$y_1^{(1)}(x,z) = \frac{1}{z}\int\limits_{\tau_2}^x q_1(t_1)\sin z(x-t_1)y_0^{(1)}(t_1,z))dt_1 + \frac{1}{z}\int\limits_{\tau_2}^x q_2(t_1)\sin z(x-t_1)y_1(t_1-\tau_2,z)dt_1$$

$$= \frac{1}{z}\int_{\tau_2}^{x} q_1(t_1)\sin z(x-t_1)\frac{1}{z}b_{s^2,\tau_2}(t_1,z)dt_1 + \frac{1}{z}\int_{\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\frac{1}{z}b_{s^2}(t_1-\tau_2,z)dt_1$$

$$= \frac{1}{z^2}b^{(1,2)}_{s^3,\tau_2}(x,z) + \frac{1}{z^2}b^{(2,1)}_{s^3,\tau_2}(x,z) = \sum_{P\in S_2(1,1)} \frac{1}{z^2}b^{P}_{s^3,\tau_2}(x,z),$$

and

$$y_2^{(1)}(x,z) = \frac{1}{z}\int_{\tau_2}^{x} q_1(t_1)\sin z(x-t_1)y_1^{(1)}(t_1,z)dt_1 + \frac{1}{z}\int_{\tau_2}^{x} q_2(t_1)\sin z(x-t_1)y_2(t_1-\tau_2,z)dt_1$$

$$= \frac{1}{z}\int_{\tau_2}^{x} q_1(t_1)\sin z(x-t_1)\left(\sum_{P\in S_2(1,1)} \frac{1}{z^2}b^{P}_{s^3,\tau_2}(t_1,z)\right)dt_1 +$$

$$+\frac{1}{z}\int_{\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\frac{1}{z^2}b_{s^3}(t_1-\tau_2,z)dt_1 =$$

$$\sum_{P\in S_3^{(1)}(2,1)} \frac{1}{z^3}b^{P}_{s^4,\tau_2}(x,z) + \frac{1}{z^3}b^{P}_{s^4,\tau_2}(x,z)|_{P\in S_3^{(2)}(2,1)} = \sum_{P\in S_3(2,1)} \frac{1}{z^3}b^{P}_{s^4,\tau_2}(x,z).$$

By the method of mathematical induction it is easily shown that functions $y_k^{(1)}(x,z)$, $k = 1, 2, ..$ have the form:

$$y_k^{(1)}(x,z) = \sum_{P\in S_{k+1}(k,1)} \frac{1}{z^{k+1}}b^{P}_{s^{k+2},\tau_2}(x,z),\ k = 1, 2, ... \qquad (14)$$

Then, from (11), (12) and (14) we get the solution within $(\tau_2, 2\tau_2]$ in the form:

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) + \sum_{k=0}^{\infty} y_k^{(1)}(x,z) = \sin zx + \frac{1}{z}b_{s^2}(x,z) + \frac{1}{z}b_{s^2,\tau_2}(x,z) +$$

$$+\sum_{k=2}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(x,z) + \sum_{k=1}^{\infty}\sum_{P\in S_{k+1}(k,1)} \frac{1}{z^{k+1}}b^{P}_{s^{k+2},\tau_2}(x,z).$$

3. From (11) for $i = 2$ we get

$$y_0^{(2)}(x,z) = \frac{1}{z}\int_{2\tau_2}^{x} q_2(t_1)\sin z(x-t_1)y_0^{(1)}(t_1-\tau_2,z)dt_1 =$$

$$= \frac{1}{z}\int\limits_{2\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\frac{1}{z}b_{s^2,\tau_2}(t_1-\tau_2,z)dt_1 = \frac{1}{z^2}b_{s^3,2\tau_2}(x,z)$$

and

$$y_1^{(2)}(x,z) = \frac{1}{z}\int\limits_{2\tau_2}^{x} q_1(t_1)\sin z(x-t_1)y_0^{(2)}(t_1,z)dt_1 + \frac{1}{z}\int\limits_{2\tau_2}^{x} q_2(t_1)\sin z(x-t_1)y_1^{(1)}(t_1-\tau_2,z)dt_1$$

$$= \frac{1}{z}\int\limits_{2\tau_2}^{x} q_1(t_1)\sin z(x-t_1)\frac{1}{z^2}b_{s^3,2\tau_2}(t_1,z)dt_1 +$$

$$+\frac{1}{z}\int\limits_{2\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\left(\sum_{P\in S_2(1,1)}\frac{1}{z^2}b^P_{s^3,\tau_2}(t_1-\tau_2,z)\right)dt_1 =$$

$$\frac{1}{z^3}b^P_{s^4,2\tau_2}(x,z)\,|_{P\in S_3^{(1)}(1,2)} + \sum_{P\in S_3^{(2)}(1,2)}\frac{1}{z^3}b^P_{s^4,2\tau_2}(x,z) = \sum_{P\in S_3(1,2)}\frac{1}{z^3}b^P_{s^4,2\tau_2}(x,z).$$

By the method of mathematical induction, we prove that

$$y_k^{(2)}(x,z) = \sum_{P\in S_{k+2}(k,2)}\frac{1}{z^{k+2}}b^P_{s^{k+3},2\tau_2}(x,z),\ k=1,2,..$$

so, the solution within the interval $(2\tau_2, 3\tau_2]$ has the form:

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) + \sum_{k=0}^{\infty} y_k^{(1)}(x,z) + \sum_{k=0}^{\infty} y_k^{(2)}(x,z) =$$

$$= \sin zx + \frac{1}{z}b_{s^2}(x,z) + \frac{1}{z}b_{s^2,\tau_2}(x,z) + \frac{1}{z^2}b_{s^3,2\tau_2}(x,z) +$$

$$+\sum_{k=2}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(x,z) + \sum_{k=1}^{\infty}\sum_{P\in S_{k+1}(k,1)}\frac{1}{z^{k+1}}b^P_{s^{k+2},\tau_2}(x,z) +$$

$$+\sum_{k=1}^{\infty}\sum_{P\in S_{k+2}(k,2)}\frac{1}{z^{k+2}}b^P_{s^{k+3},2\tau_2}(x,z) =$$

$$\sin zx+\sum_{k=1}^{2}\frac{1}{z^k}b_{s^{k+1},k\tau_2}(x,z)+\sum_{k=1}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(x,z)+\sum_{i=1}^{2}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}b^P_{s^{k+1},i\tau_2}(x,z).$$

4. Now, by the method of mathematical induction, we will prove that functions $y_k^{(i)}(x,z)$ for every $i=1,2,...k_0$ have the form:

$$y_k^{(i)}(x,z)=\sum_{P\in S_{k+i}(k,i)}\frac{1}{z^{k+i}}b^P_{s^{k+i+1},i\tau_2}(x,z),\ k=1,2... \tag{15}$$

From (14) we get that (15) is correct for $i=1$. Let us assume that (15) is valid for $i<i_0$, $1\le i_0\le k_0$, and then we show that (15) is correct for $i=i_0$. From (11), for $k=1$ and $i=i_0$, we have

$$y_1^{(i_0)}(x,z)=\int_{i_0\tau_2}^{x}q_1(t_1)\sin z(x-t_1)y_0^{(i_0)}(t_1,z)dt_1+$$

$$+\frac{1}{z}\int_{i_0\tau_2}^{x}q_2(t_1)\sin z(x-t_1)y_1^{(i_0-1)}(t_1-\tau_2,z)dt_1.$$

Taking into account that $y_{k-1}^{(i)}=0$ for $k=0$, it is obvious from (11) that functions $y_0^{(i)}(x,z)$, $i=1,2,...k_0$, have the form:

$$y_0^{(i)}(x,z)=\frac{1}{z^i}b_{s^{i+1},i\tau_2}(x,z). \tag{16}$$

Using (16) and (15) for $k=1$, $i=i_0-1$ we have

$$y_1^{(i_0)}(x,z)=\frac{1}{z}\int_{i_0\tau_2}^{x}q_1(t_1)\sin z(x-t_1)\frac{1}{z^{i_0}}b_{s^{i_0+1},i_0\tau_2}(t_1,z)dt_1$$

$$+\frac{1}{z}\int_{i_0\tau_2}^{x}q_2(t_1)\sin z(x-t_1)\left(\sum_{P\in S_{i_0}(1,i_0-1)}\frac{1}{z^{i_0}}b^P_{s^{i_0+1},(i_0-1)\tau_2}(t_1-\tau_2,z)\right)dt_1$$

$$=\frac{1}{z^{i_0+1}}b^P_{s^{i_0+2},i_0\tau_2}(x,z)\mid_{P\in S^{(1)}_{i_0+1}(1,i_0)}+\sum_{P\in S^{(2)}_{i_0+1}(1,i_0)}\frac{1}{z^{i_0+1}}b^P_{s^{i_0+2},i_0\tau_2}(x,z)$$

$$= \sum_{P\in S_{i_0+1}(1,i_0)} \frac{1}{z^{i_0+1}} b^P_{s^{i_0+2},i_0\tau_2}(x,z),$$

so (15) is correct for $k = 1$. For $y^{(i_0)}_{k+1}(x,z)$ we have

$$y^{(i_0)}_{k+1}(x,z) = \frac{1}{z}\int_{i_0\tau_2}^{x} q_1(t_1)\sin z(x-t_1) y^{(i_0)}_k(t_1,z)dt_1$$

$$+\frac{1}{z}\int_{i_0\tau_2}^{x} q_2(t_1)\sin z(x-t_1) y^{(i_0-1)}_{k+1}(t_1-\tau_2,z)dt_1$$

$$= \frac{1}{z}\int_{i_0\tau_2}^{x} q_1(t_1)\sin z(x-t_1)\left(\sum_{P\in S_{k+i_0}(k,i_0)} \frac{1}{z^{k+i_0}} b^P_{s^{k+i_0+1},i_0\tau_2}(t_1,z)\right)dt_1 +$$

$$+\frac{1}{z}\int_{i_0\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\left(\sum_{P\in S_{k+i_0}(k+1,i_0-1)} \frac{1}{z^{k+i_0}} b^P_{s^{k+i_0+1},(i_0-1)\tau_2}(t_1-\tau_2,z)\right)dt_1$$

$$= \sum_{P\in S^{(1)}_{k+i_0+1}(k+1,i_0)} \frac{1}{z^{k+i_0+1}} b^P_{s^{k+i_0+2},i_0\tau_2}(x,z) + \sum_{P\in S^{(2)}_{k+i_0+1}(k+1,i_0)} \frac{1}{z^{k+i_0+1}} b^P_{s^{k+i_0+2},i_0\tau_2}(x,z)$$

$$= \sum_{P\in S_{k+i_0+1}(k+1,i_0)} \frac{1}{z^{k+i_0+1}} b^P_{s^{k+i_0+2},i_0\tau_2}(x,z),$$

so (15) is correct for $k+1$.

5. Now we can determine the solution of the boundary value problem within the interval $(n\tau_2,(n+1)\tau_2]$, $n = 1,2,...k_0$. From (11) we have

$$y(x,z) = \sum_{k=0}^{\infty} y_k(x,z) + \sum_{i=1}^{n}\sum_{k=0}^{\infty} y^{(i)}_k(x,z).$$

and using (15), we get the solution within the interval $(n\tau_2,(n+1)\tau_2]$ in the form:

$$y(x,z) = \sin zx + \sum_{k=1}^{n} \frac{1}{z^k} b_{s^{k+1},k\tau_2}(x,z) + \sum_{k=1}^{\infty} \frac{1}{z^k} b_{s^{k+1}}(x,z) +$$

$$+\sum_{i=1}^{n}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}b^P_{s^{k+1},i\tau_2}(x,z).$$

From here for $n=k_0$, we get that the solution within the interval $(k_0\tau_2,\pi]$ has the form of (10), thus proving the theorem.

Now from (10) and (3) we will determine the characteristic function of operator L. In order to simplify, hereinafter we will write $b(z)$ instead of $b(\pi,z)$.

Theorem 2.2. *The characteristic function of the boundary value problem (1)-(3) has the form:*

$$F(z)=\sin\pi z+\frac{1}{z}\left(b_{s^2}(z)+b_{s^2,\tau_2}(z)\right)+\frac{1}{z^2}\left(b_{s^3}(z)+b_{s^3,2\tau_2}(z)+\sum_{P\in S_2(1,1)}b^P_{s^3,\tau_2}(z)\right)$$

$$+\sum_{k=3}^{k_0}\frac{1}{z^k}b_{s^{k+1},k\tau_2}(z)+\sum_{k=3}^{\infty}\frac{1}{z^k}b_{s^{k+1}}(z)+\sum_{k=2}^{\infty}\sum_{P\in S_{k+1}(k,1)}\frac{1}{z^{k+1}}b^P_{s^{k+2},\tau_2}(z)+$$

$$+\sum_{i=2}^{k_0}\sum_{k=i+1}^{\infty}\sum_{P\in S_k(k-i,i)}\frac{1}{z^k}b^P_{s^{k+1},i\tau_2}(z). \tag{17}$$

3 Asymptotic of eigenvalues

It is known that eigenvalues λ_n of the operator L are squares of zeros of the characteristic function and that zeros of the characteristic function have the form:

$$z_n=n+\aleph_n,\ \ \aleph_n\in l_2.$$

In this paper we will study the asymptotic of eigenvalues in detail because it will be the base for further consideration of the inverse problems for this class of operators by new method based on direct relations between eigenvalues and Fourier's coefficients. Because of that, we will determine the asymptotic of zeros of the characteristic function in the form:

$$z_n=n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+\frac{C_3(n)}{n^3}+o\left(\frac{1}{n^3}\right),\ (n\longrightarrow\infty) \tag{18}$$

In order to simplify, hereinafter we will write C_1, C_2, C_3 instead of $C_1(n), C_2(n), C_3(n)$, respectively. From (14), we get asymptotic of the characteristic function in the form:

$$F(z) = \sin \pi z + \frac{1}{z}\left(b_{s^2}(z) + b_{s^2,\tau_2}(z)\right) + \frac{1}{z^2}\left(b_{s^3}(z) + b_{s^3,2\tau_2}(z) + \sum_{P\in S_2(1,1)} b^P_{s^3,\tau_2}(z)\right) +$$

$$+\frac{1}{z^3}\left(b_{s^4}(z) + b_{s^4,3\tau_2}(z) + \sum_{P\in S_3(2,1)} b^P_{s^4,\tau_2}(z) \sum_{P\in S_3(1,2)} b^P_{s^4,2\tau_2}(z)\right) + o\left(\frac{b_{s^4}(z)}{z^3}\right),\ z\to\infty \tag{19}$$

Let us define so called $transitional function$ $\tilde{q}$

$$\tilde{q}(t_1) = \begin{cases} q_1(t_1), & t_1 \in [0, \frac{\tau_2}{2}) \cup (\pi - \frac{\tau_2}{2}, \pi] \\ q_1(t_1) + q_2(t_1 + \frac{\tau_2}{2}), & t_1 \in [\frac{\tau_2}{2}, \pi - \frac{\tau_2}{2}] \end{cases} \tag{20}$$

and introduce the following:

$$J_1^1 = \int_0^\pi q_1(t_1)dt_1;\ J_1^2 = \int_{\tau_2}^\pi q_2(t_1)dt_1;\ J_2^1 = \int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)dt_2dt_1;$$

$$J_2^2 = \int_{2\tau_2}^\pi q_2(t_1)\int_{\tau_2}^{t_1-\tau_2} q_2(t_2)dt_2dt_1;\ J_2^{12} = \int_{\tau_2}^\pi q_1(t_1)\int_{\tau_2}^{t_1} q_2(t_2)dt_2dt_1;$$

$$J_2^{21} = \int_{\tau_2}^\pi q_2(t_1)\int_0^{t_1-\tau_2} q_1(t_2)dt_2dt_1;\ J_3^1 = \int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)dt_3dt_2dt_1;$$

$$J_3^2 = \int_{3\tau_2}^\pi q_2(t_1)\int_{2\tau_2}^{t_1-\tau_2} q_2(t_2)\int_{\tau_2}^{t_2-\tau_2} q_2(t_3)dt_3dt_2dt_1;$$

$$J_3^{112} = \int_{\tau_2}^\pi q_1(t_1)\int_{\tau_2}^{t_1} q_1(t_2)\int_{\tau_2}^{t_2} q_2(t_3)dt_3dt_2dt_1;$$

$$J_3^{121} = \int_{\tau_2}^\pi q_1(t_1)\int_{\tau_2}^{t_1} q_2(t_2)\int_0^{t_2-\tau_2} q_1(t_3)dt_3dt_2dt_1;$$

$$J_3^{211} = \int_{\tau_2}^{\pi} q_2(t_1) \int_{0}^{t_1-\tau_2} q_1(t_2) \int_{0}^{t_2} q_1(t_3) dt_3 dt_2 dt_1;$$

$$J_3^{122} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{2\tau_2}^{t_1} q_2(t_2) \int_{t_2}^{t_2-\tau_2} q_1(t_3) dt_3 dt_2 dt_1;$$

$$J_3^{212} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{\tau_2}^{t_1-\tau_2} q_1(t_2) \int_{t_2}^{t_2} q_2(t_3) dt_3 dt_2 dt_1;$$

$$J_3^{221} = \int_{2\tau_2}^{\pi} q_2(t_1) \int_{\tau_2}^{t_1-\tau_2} q_2(t_2) \int_{0}^{t_2-\tau_2} q_1(t_3) dt_3 dt_2 dt_1;$$

$$\tilde{a}_c(z) = \int_0^{\pi} \tilde{q}(t_1) \cos z(\pi - 2t_1) dt_1; \ \tilde{a}_s(z) = \int_0^{\pi} \tilde{q}(t_1) \sin z(\pi - 2t_1) dt_1. \qquad (21)$$

Then we have

$$b_{s^2}(z) = -\frac{J_1^1}{2} cos\pi z + \frac{1}{2} \int_0^{\pi} q_1(t_1) \cos z(\pi - 2t_1) dt_1$$

$$b_{s^2,\tau_2}(z) = -\frac{J_1^2}{2} cos z(\pi - \tau_2) + \frac{1}{2} \int_{\tau_2}^{\pi} q_2(t_1) \cos z(\pi - 2t_1 + \tau_2) dt_1$$

e.i.

$$b_{s^2}(z) + b_{s^2,\tau_2}(z) = \frac{1}{2}\tilde{a}_c(z) - \frac{J_1^1}{2} cos\pi z - \frac{J_1^2}{2} cos z(\pi - \tau_2). \qquad (22)$$

Using trigonometric identities for transformation a product of trigonometric functions into a sum, we get

$$b_{s^3}(z) = \int_0^{\pi} q_1(t_1) \sin z(\pi - t_1) \int_0^{t_1} q_1(t_2) \sin z(t_1 - t_2) \sin z t_2 dt_2 dt_1 =$$

$$= -\frac{J_2^1}{4} \sin \pi z - \frac{1}{4}\beta_1^{(1)}(z) + \frac{1}{4}\beta_2^{(1)}(z) + \frac{1}{4}\beta_3^{(1)}(z)$$

$$\beta_i^{(1)}(z) = \int_0^{\pi} q_1(t_1) \int_0^{t_1} q_1(t_2) \sin z(\pi - 2t_i) dt_2 dt_1, \ i = 1, 2$$

$$\beta_3^{(1)}(z) = \int_0^{\pi} q_1(t_1) \int_0^{t_1} q_1(t_2) \sin z(\pi - 2t_1 + 2\tau_2) dt_2 dt_1.$$

Now, changing the order of integration in $\beta_2^{(1)}(z)$, and using the method of substitution of variables in $\beta_3^{(1)}(z)$, we get

$$\beta_2^{(1)}(z) = \int_0^{\pi} \left(q_1(t_1) \int_{t_1}^{\pi} q_1(t_2) dt_2 \right) \sin z(\pi - 2t_1) dt_1,$$

$$\beta_3^{(1)}(z) = -\int_0^{\pi} \left(\int_{t_1}^{\pi} q_1(t_2) q_1(t_2 - t_1) dt_2 \right) \sin z(\pi - 2t_1) dt_1,$$

so, we can present $b_{s^3}(z)$ in the form:

$$b_{s^3}(z) = -\frac{J_2^1}{4} \sin \pi z - \frac{1}{4} \int_0^{\pi} K^1(t_1, q_1(t_1)) \sin z(\pi - 2t_1) dt_1,$$

where for $t_1 \in [0, \pi]$

$$K^1(t_1, q_1(t_1)) = q_1(t_1) \int_0^{t_1} q_1(t_2) dt_2 - q_1(t_1) \int_{t_1}^{\pi} q_1(t_2) dt_2 + \int_{t_1}^{\pi} q_1(t_2) q_1(t_2 - t_1) dt_2.$$

Analogously, for $b_{s^3, 2\tau_2}(z)$ we have

$$b_{s^3, 2\tau_2}(z) = -\frac{J_2^2}{4} \sin z(\pi - 2\tau_2) - \frac{1}{4} \int_0^{\pi} K^2(t_1, q_2(t_1)) \sin z(\pi - 2\tau_1) dt_1,$$

$$K^2(t_1, q_2(t_1)) = q_2(t_1 + \tau_2) \int_{\tau_2}^{t_1} q_2(t_2) dt_2 - q_2(t_1) \int_{t_1 + \tau_2}^{\pi} q_2(t_2) dt_2 +$$

$$+\int_{t_1+\tau_2}^{\pi} q_2(t_2)q_2(t_2-t_1)dt_2, \text{ for } t_1\in[\tau_2,\pi-\tau_2]$$

$K^2(t_1,q_2(t_1))=0$ for $t_1\in[0,\tau_2)\cup(\pi-\tau_2,\pi]$

and for integrals $b^{(1,2)}_{s^3,\tau_2}(z)$ and $b^{(2,1)}_{s^3,\tau_2}(z)$ we have

$$b^{(1,2)}_{s^3,\tau_2}(z)=-\frac{J_2^{12}}{4}\sin z(\pi-\tau_2)-\frac{1}{4}\int_0^{\pi} K^{12}(t_1,q_1(t_1),q_2(t_1))\sin z(\pi-2t_1)dt_1$$

$$b^{(2,1)}_{s^3,\tau_2}(z)=-\frac{J_2^{21}}{4}\sin z(\pi-\tau_2)-\frac{1}{4}\int_0^{\pi} K^{21}(t_1,q_1(t_1),q_2(t_1))\sin z(\pi-2t_1)dt_1$$

$$K^{12}(t_1,q_1(t_1),q_2(t_1))=q_1\left(t_1+\frac{\tau_2}{2}\right)\int_{\tau_2}^{t_1+\frac{\tau_2}{2}} q_2(t_2)dt_2-q_2\left(t_1+\frac{\tau_2}{2}\right)\int_{t_1+\frac{\tau_2}{2}}^{\pi} q_1(t_2)dt_2+$$

$$+\int_{t_1+\frac{\tau_2}{2}}^{\pi} q_1(t_2)q_2(t_2-t_1-\frac{\tau_2}{2})dt_2 \quad \text{for} \quad t_1\in\left[\frac{\tau_2}{2},\pi-\frac{\tau_2}{2}\right]$$

$K^{12}(t_1,q_1(t_1)q_2(t_1))=0$ for $t_1\in[0,\frac{\tau_2}{2})\cup(\pi-\frac{\tau_2}{2},\pi]$,

$$K^{21}(t_1,q_1(t_1),q_2(t_1))=q_2\left(t_1+\frac{\tau_2}{2}\right)\int_{0}^{t_1-\frac{\tau_2}{2}} q_1(t_2)dt_2-q_1\left(t_1-\frac{\tau_2}{2}\right)\int_{t_1+\frac{\tau_2}{2}}^{\pi} q_2(t_2)dt_2+$$

$$+\int_{t_1+\frac{\tau_2}{2}}^{\pi} q_2(t_2)q_1(t_2-t_1-\frac{\tau_2}{2})dt_2 \quad \text{for} \quad t_1\in[\frac{\tau_2}{2},\pi-\frac{\tau_2}{2}],$$

$K^{21}(t_1,q_1(t_1)q_2(t_1))=0$ for $t_1\in[0,\frac{\tau_2}{2})\cup(\pi-\frac{\tau_2}{2},\pi]$.

From relations above, we get

$$b_{s^3}(z)+b_{s^3,2\tau_2}(z)+b^{(1,2)}_{s^3,\tau_2}(z)+b^{(2,1)}_{s^3,\tau_2}(z)=$$

$$= -\frac{J_2^1}{4}\sin z\pi - \frac{J_2^2}{4}\sin z(\pi - 2\tau_2) - \frac{J_2^{12}+J_2^{21}}{4}\sin z(\pi-\tau_2) - \frac{1}{4}a_s^{s^3}(z), \qquad (23)$$

where

$$a_s^{s^3}(z) = \int_0^\pi K^{s^3}(t_1, q_1(t_1), q_2(t_1))\sin z(\pi - 2t_1)dt_1,$$

$$K^{s^3}(t_1, q_1(t_1), q_2(t_1)) = K^1(t_1, q_1(t_1)) + K^2(t_1, q_2(t_1)) +$$
$$+K^{12}(t_1, q_1(t_1), q_2(t_1)) + K^{21}(t_1, q_1(t_1), q_2(t_1)),\ t_1 \in [0,\pi].$$

For the integral $b_{s^4}(z)$ we have

$$b_{s^4}(z) = \int_0^\pi q_1(t_1)\sin z(\pi - t_1)\int_0^{t_1} q_1(t_2)\sin z(t_1 - t_2)\int_0^{t_2} q_1(t_3)\sin z(t_2 - t_3)\sin zt_3 dt_3 dt_2 dt_1$$

$$= \frac{\cos \pi z}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)dt_3dt_2dt_1 -$$

$$-\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_1)dt_3dt_2dt_1 +$$

$$+\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_2)dt_3dt_2dt_1 -$$

$$-\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_3)dt_3dt_2dt_1 -$$

$$-\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_1 + 2t_2)dt_3dt_2dt_1 +$$

$$+\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_1 + 2t_3)dt_3dt_2dt_1 -$$

$$-\frac{1}{8}\int_0^\pi q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi - 2t_2 + 2t_3)dt_3dt_2dt_1 +$$

$$+\frac{1}{8}\int_0^{\pi} q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)\cos z(\pi-2t_1+2t_2-2t_3)dt_3dt_2dt_1 \qquad (24)$$

Changing the order of integration and/or using the method of substitution of variables in last seven integrals in (24), we get

$$b_{s^4}(z)=\frac{J_3^1}{8}\cos \pi z+\frac{1}{8}\int_0^{\pi} T^1(t_1,q_1(t_1))\cos z(\pi-2t_1)dt_1$$

$$T^1(t_1,q_1(t_1))=-q_1(t_1)\int_0^{t_1} q_1(t_2)\int_0^{t_2} q_1(t_3)dt_3dt_2+q_1(t_1)\int_{t_1}^{\pi} q_1(t_2)\int_0^{t_1} q_1(t_3)dt_3dt_2$$

$$-q_1(t_1)\int_{t_1}^{\pi} q_1(t_3)\int_{t_1}^{t_3} q_1(t_2)dt_3dt_2+\int_{t_1}^{\pi} q_1(t_2)q_1(t_2-t_1)\int_0^{t_2-t_1} q_1(t_3)dt_3dt_2$$

$$-\int_{t_1}^{\pi} q_1(t_3)q_1(t_3-t_1)\int_{t_3-t_1}^{\pi} q_1(t_2)dt_3dt_2+\int_{t_1}^{\pi}\int_{t_1}^{t_3} q_1(t_3)q_1(t_2)q_1(t_2-t_3)dt_3dt_2-$$

$$-q_1(t_1)\int_{t_1}^{\pi}\int_{t_3-t_1}^{\pi} q_1(t_2)q_1(t_1-t_3+t_2)dt_3dt_2.$$

Now, in the similar way, we can detrmine functions $T^2(t_1,q_2(t_1))$ and $T^p(t_1,q_1(t_1),q_2(t_1))$, $P\in S_3(2,1)\cup S_3(1,2)$ with characteristics:

$$b_{s^4,3\tau_2}(z)=\frac{J_3^2}{8}\cos z(\pi-3\tau_2)+\frac{1}{8}\int_0^{\pi} T^2(t_1,q_2(t_1))\cos z(\pi-2t_1)dt_1$$

$$b_{s^4,\tau_2}(z)=\frac{J_3^P}{8}\cos z(\pi-\tau_2)+\frac{1}{8}\int_0^{\pi} T^P(t_1,q_1(t_1),q_2(t_1))\cos z(\pi-2t_1)dt_1,\ P\in S_3(2,1)$$

$$b_{s^4,2\tau_2}(z)=\frac{J_3^P}{8}\cos z(\pi-2\tau_2)+\frac{1}{8}\int_0^{\pi} T^P(t_1,q_1(t_1),q_2(t_1))\cos z(\pi-2t_1)dt_1,\ P\in S_3(1,2).$$

From relations above, we get

$$b_{s^4}(z)+b_{s^4,3\tau_2}(z)+\sum_{P\in S_3(2,1)} b^P_{s^4,\tau_2}(z)+\sum_{P\in S_3(1,2)} b^P_{s^4,2\tau_2}(z)=$$

$$=\frac{J_3^1}{8}\cos\pi z+\frac{J_3^{211}+J_3^{121}+J_3^{112}}{8}\cos z(\pi-\tau_2)+$$

$$+\frac{J_3^{221}+J_3^{122}+J_3^{212}}{8}\cos z(\pi-2\tau_2)+\frac{J_3^2}{8}\cos z(\pi-3\tau_2)+\frac{1}{8}a_c^{s^4}(z) \qquad (25)$$

where

$$a_c^{s^4}(z)=\int_0^\pi T^{s^4}(t_1,q_1(t_1),q_2(t_1))\cos z(\pi-2t_1)dt_1,$$

and the function $T^{s^4}(t_1,q_1(t_1),q_2(t_1))$ is the sum of functions $T^1(t_1,q_1(t_1))$, $T^2(t_1,q_2(t_1))$ and $T^P(t_1,q_1(t_1),q_2(t_1))$, $P\in S_3(2,1)\cup S_3(1,2)$.
Now, substituting realations (22), (23) and (25)into (19), we get

$$F(z)=\sin\pi z-\frac{1}{z}\left[\frac{J_1^1}{2}\cos z\pi+\frac{J_1^2}{2}\cos z(\pi-\tau_2)-\frac{1}{2}\tilde{a}_c(z)\right]-$$

$$-\frac{1}{z^2}\left[\frac{J_2^1}{4}\sin\pi z+\frac{J_2^{12}+J_2^{21}}{4}\sin z(\pi-\tau_2)+\frac{J_2^2}{4}\sin z(\pi-2\tau_2)+\frac{1}{4}a_s^{s^3}(z)\right]+$$

$$+\frac{1}{z^3}\left[\frac{J_3^1}{8}\cos\pi z+\frac{J_3^{211}+J_3^{121}+J_3^{112}}{8}\cos z(\pi-\tau_2)\right]+$$

$$+\frac{1}{z^3}\left[\frac{J_3^{221}+J_3^{122}+J_3^{212}}{8}\cos z(\pi-2\tau_2)+\frac{J_3^2}{8}\cos z(\pi-3\tau_2)\right]+o\left(\frac{1}{z^3}\right) \qquad (26)$$

Further, from (18) we get

$$\sin\pi z_n=(-1)^n\left(\frac{\pi C_1}{n}+\frac{\pi C_2}{n^2}+\frac{6\pi C_3-\pi^3C_1^3}{6n^3}\right)+o\left(\frac{1}{n^3}\right),$$

$$\cos\pi z_n=(-1)^n\left(1-\frac{\pi^2C_1^2}{2n^2}\right)+o\left(\frac{1}{n^2}\right);\ \frac{1}{z_n}=\frac{1}{n}-\frac{C_1}{n^3}+o\left(\frac{1}{n^3}\right),$$

$$\cos z_n(\pi-\tau_2)=(-1)^n\left(1-\frac{(\pi-\tau_2)^2C_1^2}{2n^2}\right)\cos n\tau_2+$$

$$+(-1)^n\left(\frac{(\pi-\tau_2)C_1}{n}+\frac{(\pi-\tau_2)C_2}{n^2}\right)\sin n\tau_2+o\left(\frac{1}{n^2}\right),$$

$$\sin z_n(\pi-\tau_2)=(-1)^{n+1}\sin n\tau_2+(-1)^n\frac{(\pi-\tau_2)C_1}{n}\cos n\tau_2+o\left(\frac{1}{n}\right),$$

$$\tilde{a}_c(z_n)=(-1)^n\tilde{a}_{2n}+(-1)^n\pi\frac{C_1}{n}\tilde{b}_{2n}-2(-1)^n\frac{C_1}{n}\tilde{b}^*_{2n}+o\left(\frac{1}{n^2}\right),$$

$$a_s^{s^3}(z_n)=(-1)^{n+1}b_{2n}^{s^3}+o\left(\frac{1}{n}\right), \tag{27}$$

where

$$\tilde{a}_{2n}=\int\limits_0^\pi\tilde{q}(t_1)\cos 2nt_1dt_1;\ \tilde{b}_{2n}=\int\limits_0^\pi\tilde{q}(t_1)\sin 2nt_1dt_1;\ \tilde{b}^*_{2n}=\int\limits_0^\pi t\tilde{q}(t_1)\sin 2nt_1dt_1.$$

$$b_{2n}^{s^3}=\int\limits_0^\pi K^{s^3}(t_1,q_1(t_1),q_2(t_1))\sin 2nt_1dt_1.$$

Inserting asymptotic relations (27) into (26), we get

$$F(z_n)=(-1)^n\left(\frac{\pi C_1}{n}+\frac{\pi C_2}{n^2}+\frac{6\pi C_3-\pi^3C_1^3}{6n^3}\right)-(-1)^n\frac{J_1^1}{2}\left(\frac{1}{n}-\frac{C_1}{n^3}\right)\left(1-\frac{\pi^2C_1^2}{2n^2}\right)-$$

$$\frac{(-1)^nJ_1^2}{2}\left(\frac{1}{n}-\frac{C_1}{n^3}\right)\left[\left(1-\frac{(\pi-\tau_2)^2C_1^2}{2n^2}\right)\cos n\tau_2+\left(\frac{(\pi-\tau_2)C_1}{n}+\frac{(\pi-\tau_2)C_2}{n^2}\right)\sin n\tau_2\right]$$

$$-\frac{(-1)^n}{2}\left(\frac{1}{n}-\frac{C_1}{n^3}\right)\left[\tilde{a}_{2n}+\pi\frac{C_1}{n}\tilde{b}_{2n}-2\frac{C_1}{n}\tilde{b}^*_{2n}\right]-\frac{(-1)^n\pi C_1}{4n^3}J_2^1-$$

$$-\frac{(-1)^n}{4n^2}\left(-\sin n\tau_2+\frac{(\pi-\tau_2)C_1}{n}\cos n\tau_2\right)(J_2^{12}+J_2^{21})-\frac{(-1)^{n+1}}{4n^2}b_{2n}^{s^3}$$

$$-\frac{(-1)^n}{4n^2}\left(-\sin 2n\tau_2+\frac{(\pi-2\tau_2)C_1}{n}\cos 2n\tau_2\right)J_2^2+\frac{(-1)^n}{8n^3}(J_3^{211}+J_3^{121}+J_3^{112})\cos n\tau_2+$$

$$+\frac{(-1)^n}{8n^3}\left[J_3^1+(J_3^{221}+J_3^{122}+J_3^{212})\cos 2n\tau_2+J_3^2\cos 3n\tau_2\right]+o\left(\frac{1}{n^3}\right) \tag{28}$$

Now, since $F(z_n)=0$, from (28) we get the system of equations:

$$\pi C_1-\frac{J_1^1}{2}-\frac{J_1^2}{2}\cos n\tau_2+\frac{1}{2}\tilde{a}_{2n}=0,$$

$$\pi C_2-\frac{J_1^1}{2}(\pi-\tau_2)C_1\sin n\tau_2+\frac{\pi C_1}{2}\tilde{b}_{2n}-C_1\tilde{b}^*_{2n}+\frac{J_2^{12}+J_2^{21}}{4}\sin n\tau_2+\frac{J_2^2}{4}\sin 2n\tau_2+\frac{1}{4}b^{s^3}_{2n}=0,$$

$$\pi C_3-\frac{\pi^3C_1^3}{6}+\frac{\pi^2C_1^2}{4}J_1^1+\frac{C_1J_1^1}{2}+\frac{(\pi-\tau_2)^2C_1^2}{4}J_1^2\cos n\tau_2-\frac{J_2^1}{4}\pi C_1+\frac{C_1J_1^2}{2}\cos n\tau_2+\frac{C_1\tilde{a}_{2n}}{2}-$$

$$-\frac{J_1^2}{2}(\pi-\tau_2)C_2\sin n\tau_2-\frac{J_2^{12}+J_2^{21}}{4}(\pi-\tau_2)C_1\cos n\tau_2-\frac{J_2^2}{4}(\pi-2\tau_2)C_1\cos 2n\tau_2+$$

$$\frac{1}{8}[J_3^1+(J_3^{211}+J_3^{121}+J_3^{112})\cos n\tau_2+(J_3^{221}+J_3^{122}+J_3^{212})\cos 2n\tau_2+J_3^2\cos 3n\tau_2]=0 \qquad (29)$$

From the first equation of the system (29), we get

$$C_1=\frac{J_1^1}{2\pi}+\frac{J_1^2}{2\pi}\cos n\tau_2-\frac{\tilde{a}_{2n}}{2\pi} \qquad (30)$$

Fom the second equation of the system (29), we get

$$C_2=\left(\frac{\pi-\tau_2}{2\pi}J_1^2\sin n\tau_2-\frac{\tilde{b}_{2n}}{2}+\frac{\tilde{b}^*_{2n}}{\pi}\right)C_1-\frac{J_2^{12}+J_2^{21}}{4\pi}\sin n\tau_2-\frac{J_2^2}{4\pi}\sin 2n\tau_2-\frac{1}{4\pi}b^{s^3}_{2n},$$

or

$$C_2=l_0\tilde{b}_{2n}+l_1\sin n\tau_2+l_2\sin 2n\tau_2+l_3 \qquad (31)$$

where

$$l_0=-\frac{J_1^1}{4\pi}-\frac{J_1^2}{4\pi}\cos n\tau_2;\ l_1=\frac{\pi-\tau_2}{4\pi^2}J_1^1J_1^2-\frac{J_2^{12}+J_2^{21}}{4\pi}-\frac{\pi-\tau_2}{4\pi^2}J_1^2\tilde{a}_{2n},$$

$$l_2=\frac{\pi-\tau_2}{8\pi^2}(J_1^2)^2-\frac{J_2^2}{4\pi};\ l_3=\tilde{b}^*_{2n}\left(\frac{J_1^1}{2\pi^2}+\frac{J_1^2}{2\pi^2}\cos n\tau_2-\frac{\tilde{a}_{2n}}{2\pi^2}\right)+\frac{\tilde{b}_{2n}\tilde{a}_{2n}}{4\pi}-\frac{1}{4\pi}b^{s^3}_{2n}. \qquad (32)$$

From the third equation of the system (29), we get

$$C_3=\frac{\pi^2}{6}C_1^3-\left(\frac{\pi}{4}J_1^1+\frac{(\pi-\tau_2)^2}{4\pi}J_1^2\cos n\tau_2\right)C_1^2+\frac{\pi-\tau_2}{2\pi}J_1^2\sin n\tau_2C_2-$$

$$\left(\frac{J_2^1}{4}+\frac{J_1^1}{2\pi}+\frac{(\pi-\tau_2)(J_2^{12}+J_2^{21})+2J_1^2}{4\pi}\cos n\tau_2+\frac{\pi-2\tau_2}{4\pi}J_2^2\cos 2n\tau_2\right)C_1-$$

$$\frac{1}{8\pi}[J_3^1+(J_3^{211}+J_3^{121}+J_3^{112})\cos n\tau_2+(J_3^{221}+J_3^{122}+J_3^{212})\cos 2n\tau_2+J_3^2\cos 3n\tau_2]+o(1). \qquad (33)$$

Using trigonometric identities

$$\cos^3 x=\frac{1}{4}(\cos 3x+3\cos x),\ \cos^2 x=\frac{1}{2}(1+\cos 2x),\ \sin^2 x=\frac{1}{2}(1-\cos 2x)$$

$$\sin x\sin 2x=\frac{1}{2}(\cos x-\cos 3x),\ \cos x\cos 2x=\frac{1}{2}(\cos x+\cos 3x)$$

from (30) we get

$$C_1^2=\frac{(J_1^1)^2}{4\pi^2}+\frac{(J_1^2)^2}{8\pi^2}+\frac{J_1^1J_1^2}{2\pi^2}\cos n\tau_2+\frac{(J_1^2)^2}{8\pi^2}\cos 2n\tau_2+o(1)$$

$$C_1^3=\frac{(J_1^1)^3}{8\pi^3}+\frac{3J_1^1(J_1^2)^2}{16\pi^3}+\left(\frac{3J_1^2(J_1^1)^2}{8\pi^3}+\frac{3(J_1^2)^3}{32\pi^3}\right)\cos n\tau_2+$$
$$+\frac{3J_1^1(J_1^2)^2}{16\pi^3}\cos 2n\tau_2+\frac{(J_1^2)^3}{32\pi^3}\cos 3n\tau_2+o(1) \tag{34}$$

Substituting (30), (31), (32) and (34) into (33), we get

$$C_3=d_0+d_1\cos n\tau_2+d_2\cos 2n\tau_2+d_3\cos 3n\tau_2+o(1) \tag{35}$$

where

$$d_0=-\frac{(J_1^1)^3}{24\pi}+\frac{J_1^1J_2^1}{8\pi}-\frac{(J_1^1)^2}{4\pi^2}-\frac{(J_1^2)^2}{8\pi^2}-\frac{J_3^1}{8\pi},$$

$$d_1=\frac{\pi^2-(\pi-\tau_2)^2}{64\pi^3}(J_1^2)^3-\frac{\pi^2+(\pi-\tau_2)^2}{16\pi^3}J_1^2(J_1^1)^2+\frac{J_1^2J_2^1}{8\pi}+$$
$$+\frac{(\pi-\tau_2)(J_2^{12}+J_2^{21})}{8\pi^2}J_1^1-\frac{\tau_2}{16\pi^2}J_1^2J_2^2-\frac{J_1^1J_1^2}{2\pi^2}-\frac{(J_3^{211}+J_3^{121}+J_3^{112})}{8\pi},$$

$$d_2=-\frac{(\pi-\tau_2)^2}{8\pi^3}J_1^1(J_1^2)^2+\frac{(\pi-\tau_2)(J_2^{12}+J_2^{21})}{8\pi^2}J_1^2+\frac{\pi-2\tau_2}{8\pi^2}J_1^2J_2^2-$$
$$-\frac{(J_1^2)^2}{8\pi^2}-\frac{(J_3^{221}+J_3^{122}+J_3^{212})}{8\pi}$$

$$d_3=\frac{\pi^2-9(\pi-\tau_2)^2}{192\pi^3}(J_1^2)^3+\frac{2\pi-3\tau_2}{16\pi^2}J_1^2J_2^2-\frac{J_3^2}{8\pi}. \tag{36}$$

Now, we can prove the theorem about the asymptotic of eigenvalues of the operator L.

Theorem 3.1. *If $q_i\in L_2[0,\pi],\ i=1,2,$ then the asymptotic of eigenvalues λ_n of the operator L have the representation in the form*

$$\lambda_n=z_n^2=n^2+r_0+r_1\cos n\tau_2-\frac{\tilde a_{2n}}{\pi}+r_2\frac{\tilde b_{2n}}{n}+r_3\frac{\sin n\tau_2}{n}+r_4\frac{\sin 2n\tau_2}{n}+\frac{r_5}{n}+$$
$$+\frac{r_6}{n^2}+r_7\frac{\cos n\tau_2}{n^2}+r_8\frac{\cos 2n\tau_2}{n^2}+r_9\frac{\cos 3n\tau_2}{n^2}+o(\frac{1}{n^2}) \tag{37}$$

with the following coefficients of representation

$$r_0 = \frac{J_1^1}{\pi};\ r_1 = \frac{J_1^2}{\pi};\ r_2 = -\frac{J_1^1}{2\pi} - \frac{J_1^2}{2\pi}\cos n\tau_2;$$

$$r_3 = \frac{\pi-\tau_2}{2\pi^2}J_1^1J_1^2 - \frac{J_2^{12}+J_2^{21}}{2\pi} - \frac{\pi-\tau_2}{2\pi^2}J_1^2\tilde{a}_{2n},$$

$$r_4 = \frac{\pi-\tau_2}{4\pi^2}(J_1^2)^2 - \frac{J_2^2}{2\pi};\ r_5 = \frac{\tilde{b}_{2n}^*}{\pi^2}(J_1^1 + J_1^2\cos n\tau_2 - \tilde{a}_{2n}) + \frac{\tilde{b}_{2n}\tilde{a}_{2n}}{2\pi} - \frac{1}{2\pi}b_{2n}^{s^3},$$

$$r_6 = -\frac{(J_1^1)^3}{12\pi} + \frac{J_1^1J_2^1}{4\pi} - \frac{(J_1^1)^2}{4\pi^2} - \frac{(J_1^2)^2}{8\pi^2} - \frac{J_3^1}{4\pi},$$

$$r_7 = \frac{\pi^2-(\pi-\tau_2)^2}{32\pi^3}(J_1^2)^3 - \frac{\pi^2+(\pi-\tau_2)^2}{8\pi^3}(J_1^1)^2J_1^2 + \frac{J_1^2J_2^1}{4\pi} +$$
$$+\frac{\pi-\tau_2}{4\pi^2}J_1^1(J_2^{12}+J_2^{21}) - \frac{\tau_2}{8\pi^2}J_2^2J_1^2 - \frac{J_1^1J_1^2}{2\pi^2} - \frac{J_3^{211}+J_3^{121}+J_3^{112}}{4\pi},$$

$$r_8 = -\frac{(J_1^2)^2}{4\pi^2} - \frac{(\pi-\tau_2)^2}{4\pi^3}(J_1^2)^2J_1^1 + \frac{\pi-\tau_2}{4\pi^2}J_1^2(J_1^{12}+J_2^{21}) + \frac{\pi-2\tau_2}{4\pi^2}J_2^2J_1^1 -$$
$$-\frac{J_3^{221}+J_3^{122}+J_3^{212}}{4\pi},$$

$$r_9 = \frac{\pi^2-9(\pi-\tau_2)^2}{96\pi^3}(J_1^2)^3 + \frac{2\pi-3\tau_2}{8\pi^2}J_2^2J_1^2 - \frac{J_3^2}{4\pi}. \tag{38}$$

Proof 3.1. From

$$z_n = \pm\left[n + \frac{C_1}{n} + \frac{C_2}{n^2} + \frac{C_3}{n^3} + o\left(\frac{1}{n^3}\right)\right]$$

we get

$$z_n^2 = n^2 + 2C_1 + \frac{2C_2}{n} + \frac{2C_3 + C_1^2}{n^2} + o\left(\frac{1}{n^2}\right)$$

or

$$z_n^2 = n^2 + \frac{J_1^1}{\pi} + \frac{J_1^2}{\pi}\cos n\tau_2 - \frac{\tilde{a}_{2n}}{\pi} + \frac{2}{n}(l_0\tilde{b}_{2n} + l_1\sin n\tau_2 + l_2\sin 2n\tau_2 + l_3) +$$
$$+\left(\frac{(J_1^1)^2}{4\pi^2} + \frac{(J_1^2)^2}{8\pi^2} + 2d_0\right)\frac{1}{n^2} + \left(\frac{J_1^1J_1^2}{2\pi^2} + 2d_1\right)\frac{\cos n\tau_2}{n^2} +$$
$$+\left(\frac{(J_1^2)^2}{8\pi^2} + 2d_2\right)\frac{\cos 2n\tau_2}{n^2} + 2d_3\frac{\cos 3n\tau_2}{n^2} + o(\frac{1}{n^2}). \tag{39}$$

Inserting (32) and (36) in (39), we get coefficients of representation from (38), thus proving the theorem.

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FOURTH MATHEMATICAL CONFERENCE OF THE REPUBLIC OF SRPSKA
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On a Fixed Point Theorem of Rezapour and Hamlbarani

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Abstract

I. D. Aranđelović and V. Mišić [2] introduced the notion of a contractive linear operator on metric linear spaces. In [1] authors consider contractive linear operators on locally convex topological vector spaces. General theory of contractive bounded linear operators on partial ordered (non-necessarily locally convex) Haussdorff topological vector spaces and theirs basic properties was presented in [3].

In this talk (paper) we present one common fixed point theorem with operator contractive condition which generalize some earlier result obtained by Sh. Rezapour and R. Hamlbarani [6] - Theorem 2.8.

1 Introduction

There have been a number of generalizations of metric space. One such generalization is the notion of a TVS-cone metric space initiated by I. Beg, A. Azam and M. Arshad [5]. I. D. Aranđelović and V. Mišić [2] introduced the F - cone metric spaces and the notion of a contractive linear operator and present some fixed point results with operator contractive condition. In [1] authors consider contractive linear operators on locally convex topological vector spaces. General theory of contractive bounded linear operators on partial ordered (non-necessarily locally convex) Haussdorff topological vector spaces and theirs basic properties was presented in [3].

In this talk (paper) we present one common fixed point theorem with operator contractive condition which generalize some earlier results obtained by Sh. Rezapour and R. Hamlbarani [6] - Theorem 2.8.

2 Preliminary Notes

Let E be a linear topological space. Let E be a linear topological space. A subset P of E is called a cone if:

1) P is closed, nonempty and $P \neq \{0\}$;

2) $a, b \in \mathbf{R}$, $a, b > 0$, and $x, y \in P$ imply $ax + by \in P$;

3) $P \cap (-P) = \{0\}$.

Given a cone, $P \subseteq E$ we define partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of P).

Let E be a linear topological space and let $P \subseteq E$ be a cone. We say that P is a solid cone if and only if $\text{int}P \neq \emptyset$. Then c is an interior point of P if and only if $[-c, c]$ is a neighborhood of Θ in E.

Let E be a topological vector space and $P \subseteq E$ be a cone. P is a solid cone if and only if $\text{int}P \neq \emptyset$.

In paper [3] we introduced the notion of a contractive operator by the following way.

Definition 2.1. ([3]) If $A : E \to E$ is a one to one function such that $A(P) = P$, (I-A) is one to one and $(I - A)(P) = P$ then A is contractive operator.

Basic properties of contractive bounded linear operator we present in [3]

Recently I. Beg, A. Azam and M. Arshad [5] introduced the notion of TVS-cone metric spaces, such that distance function take values Hausdorff (not necessarily locally convex) topological vector space.

In the following, we always suppose that E is a real (not necessarily locally convex) Hausdorff topological vector space, P is a solid cone in E such that $\leq$ is a partial ordering on E with respect to P. By I we denote the identity operator on E i.e. $I(x) = x$ for each $x \in E$.

Definition 2.2. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:

1) $\Theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \Theta$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a TVS-cone metric on X and (X, d) is called a TVS-cone metric space.

Definition 2.3. Let (X, d) be a solid TVS-cone metric space, $x \in X$ and (x_n) a sequence in X. Then

1) (x_n) TVS-cone converges to x if for every $c \in \text{int}P$ there exists a positive integer N such that for all $n \geq N$ $d(x_n, x) \ll c$. We denote this by $\lim x_n = x$ or $x_n \to x$;

2) (x_n) is a TVS-cone Cauchy sequences if for every $c \in \text{int}P$ there exists a positive integer N such that for all $m, n \geq N$ $d(x_m, x_n) \ll c$;

3) (X, d) is a TVS-cone complete cone metric space if every Cauchy sequence is convergent.

Lemma 2.1. *([4]) Let (X, d) be a TVS-cone metric space, $(x_n) \subseteq X$ and $A : E \to E$ a contractive bounded linear operator. If*

$$d(x_{n+1}, x_{n+2}) \leq A(d(x_n, x_{n+1})) \tag{1}$$

for any n, then (x_n) is a Cauchy sequence.

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. The element $x \in X$ is a fixed point for f if $x = f(x)$.

Let X be a nonempty set and $f : X \to X$ an arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that the sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

3 Results

Next Theorem generalizes Theorem 2.8 of [6].

Theorem 3.1. *Let (X, d) be a complete cone metric space, $f : X \to X$ and $A, B : E \to E$ contractive bounded linear operators. If for any $x, y \in X$ there exists*

$$u \in \{d(x, y), d(x, f(x)), d(y, f(y))\} \tag{2}$$

such that

$$d(f(x), f(y)) \leq A(u) + B(d(y, f(x))), \tag{3}$$

then f has a fixed point in X. Also, the fixed point of f is unique whenever $I - A - B$ is contractive operator, and for each $x \in X$ sequence of Picard iterates defined by f at x converge to the fixed point.

Proof: Let $x_0 \in X$ be arbitrary and (x_n) the sequence of Picard iterates of f at point x_0. By (2) and (3) we get that there exists

$$u \in \{d(x_0, x_1), d(x_0, x_1)), d(x_1, x_2))\}$$

such that

$$d(x_1, x_2) \leq A(u) + d(x_1, x_1) = A(u).$$

So (2) and (3) implies that

$$d(x_1, x_2)) \leq A(d(x_0, x_1)). \tag{4}$$

From (4) by induction we obtained

$$d(x_{n+1}, x_{n+2})) \leq A(d(x_n, x_{n+1}).$$

By Lemma 2.1 it follows that (x_n) is a Cauchy sequence. (x_n) is convergent because (X, d) is complete.

Let $\lim x_n = y$. Suppose that $y \neq f(y)$. Then $0 << d(y, f(y))$. Then there exists positive integer n_0 such that $n > n_0$ implies $d(x_n, y) << \dfrac{A(d(y, f(y)))}{4}$. Let $n > n_0$. Then there exists

$$u \in \{d(x_n, y), d(x_n, x_{n+1}), d(y, f(y))\}$$

such that

$$d(y, f(y)) \leq d(f(y), x_{n+1}) + d(x_{n+1}, y) \leq A(u) + B(d(y, x_{n+1})) + d(x_{n+1}, y).$$

It follows that

$$d(y, f(y)) \leq A(d(y, f(y)) + B(d(y, x_{n+1})) + d(x_n, y) << A(d(y, f(y)),$$

or

$$d(y, f(y)) \leq A(d(x_n, x_{n+1})) + B(d(y, x_{n+1})) + d(x_n, y) << A(d(y, f(y))),$$

which implies $y = Ty$.

Let $z \in X$, $z \neq y$ and $z = f(z)$. From (16) it follows

$$d(z, y) = d(f(z), f(y)) \leq A(d(z, y)) + B(d(z, f(y)) \leq (A + B)(d(z, y)).$$

So $0 \leq ((A + B) - I)(d(z, y))$. It follows $d(z, y) = 0$ which is a contradiction.$\diamondsuit$

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Totalization of the *Riemann* integral

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Abstract

The aim of this note is to define the total value of the *Riemann* integral that can be used to generalize the well-known *Newton-Leibniz* formula.

1 Introduction

As well known, by means of integral calculus it became possible to solve by a unified method many theoretical and applied problems, both new ones which earlier had not been amenable to solution, and old ones that had previously required special artificial techniques. The basic notions of integral calculus are two closely related notions of the integral, the indefinite and the definite integral. Let $[a, b]$ be some compact interval in R. It is an old result that for any *Riemann* integrable function $f : [a, b] \mapsto R$ with a primitive $F : [a, b] \mapsto R$ that is differentiable on $[a, b]$, the *Newton-Leibniz* formula holds (see [1]).

$$F(b) - F(a) = \int_a^b f(x)\, dx \tag{1}$$

This result, sometimes called the second fundamental theorem of calculus, is that the definite integral of a function can be computed by using any one of its infinitely many antiderivatives. As the cornerstone of calculus, it has key practical applications because it markedly simplifies the computation of definite integrals.

The aim of this note is to define the total value of the *Riemann* integral that can be used to extend the above mentioned result to any real valued function f that has a primitive F defined and differentiable on $[a, b]\backslash E$, where E is a certain subset of $[a, b]$ at whose points F can take values $\pm\infty$ or not be defined at all. Unless otherwise stated in what follows, we assume that the endpoints of $[a, b]$ do not belong to E.

Define point functions $F_{ex} : [a, b] \mapsto R$ and $D_{ex}F : [a, b] \mapsto R$ by extending F and its derivative f from $[a, b]\backslash E$ to E by $F_{ex}(x) = 0$ and $D_{ex}F(x) = 0$ for $x \in E$ (see [6]), so that

$$F_{ex}\left(x\right)=\begin{cases} F\left(x\right), \text{ if } x\in[a,b]\backslash E \\ 0, \text{ if } x\in E \end{cases},$$

$$D_{ex}F(x)=\begin{cases} f\left(x\right), \text{ if } x\in[a,b]\backslash E \\ 0, \text{ if } x\in E \end{cases}. \tag{2}$$

2 Preliminaries

A partition $P[a,b]$ of a compact interval $[a,b]\in R$ is a finite set (collection) of interval-point pairs $([a_i,b_i],x_i)_{i\leq\nu}$, such that the subintervals $[a_i,b_i]$ are non-overlapping, $\cup_{i\leq\nu}[a_i,b_i]=[a,b]$ and $x_i\in[a_i,b_i]$. The points $\{x_i\}_{i\leq\nu}$ are the tags of $P[a,b]$, [4]. It is evident that a given partition of $[a,b]$ can be tagged in infinitely many ways by choosing different points as tags. If E is a subset of $[a,b]$, then the restriction of $P[a,b]$ to E is a finite collection of $([a_i,b_i],x_i)\in P[a,b]$, such that each pair of sets $[a_i,b_i]$ and E intersects in at least one point and all x_i are tagged in E. In symbols, $P[a,b]|_E=\{([a_i,b_i],x_i)\in P[a,b]\mid[a_i,b_i]\cap E\neq\emptyset$ and $x_i\in E\}$. Let $\mathcal{P}[a,b]$ be the family of all partitions $P[a,b]$ of $[a,b]$. Given $\delta:[a,b]\mapsto R_+$, named a gauge, a point-interval pair $([a_i,b_i],x_i)$ is called δ-fine if $[a_i,b_i]\subseteq(x_i-\delta(x_i),x_i+\delta(x_i))$. The collection $\mathcal{I}([a,b])$ is the family of all compact subintervals I of $[a,b]\in R$. The *Lebesgue* measure of the interval I is denoted by $|I|$. Any real valued function φ defined on $\mathcal{I}([a,b])$ is an interval function, [2]. For a function $F:[a,b]\mapsto R$ the associated interval function of F is an interval function $\Delta F:\mathcal{I}([a,b])\mapsto R$, such that $F(I)=F(v)-F(u)$, where u and v are the endpoints of I. In what follows we will use the following notations

$$\Xi_f(P[a,b])=\sum_{i\leq\nu}f\left(x_i\right)\left|[a_i,b_i]\right| \text{ and}$$

$$\Sigma_{\varphi\Delta F}(P[a,b])=\sum_{i\leq\nu}\varphi([a_i,b_i])\Delta F([a_i,b_i]). \tag{3}$$

Definition 2.1. For $E\subset[a,b]$ let $D_{ex}F(x):[a,b]\mapsto R$ be defined by (2). Then, the point function f is said to be *Riemann* integrable to a real number A on $[a,b]$ if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon(x)\equiv\underline{\delta}_\varepsilon=inf\left\{\delta_\varepsilon(x)\mid x\in[a,b]\right\}>0$ such that $|\Xi_{D_{ex}F}(P[a,b])-A|<\varepsilon$, whenever $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-fine partition. In symbols, $A=vp\int_a^b f\left(x\right)dx$.

Definition 2.2. Let $\phi:I([a,b])\mapsto R$ and $E\subseteq[a,b]$. A function $f:[a,b]\mapsto R$ is the limit of ϕ on $[a,b]\backslash E$ if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon\equiv\underline{\delta}_\varepsilon$ such that

$$\left|\phi\left([a_i,b_i]\right)-f\left(x_i\right)\right|<\varepsilon,$$

whenever $([a_i,b_i],x_i)\in P[a,b]\backslash P[a,b]|_E$ and $P[a,b]\in\mathcal{P}[a,b]$ is a δ_ε-fine partition.

Definition 2.3. Let $F : [a, b] \mapsto R$ and let $f : [a, b] \mapsto R$. Then, F is said to be differentiable to f on $[a, b]$, if f is the limit of φ on $[a, b]$, defined by

$$\varphi(I) = \Delta F(I)/\Delta x(I), \tag{4}$$

where $\Delta x(I) = |I|$ and $I \in \mathcal{I}([a, b])$.

3 Main results

For a compact set $[a, b] \in R$ let E be a certain subset of $[a, b]$, such that a point function F is defined and differentiable on $[a, b]\backslash E$. If f is the limit of φ_{ex} $= \Delta F_{ex}/\Delta x$ on $[a, b]\backslash E$, then for every $\varepsilon > 0$ we can define a set

$$\Gamma_\varepsilon = \{(x, I) \mid x \in [a, b] \text{ is a point of } I \in \mathcal{I}([a, b]) \text{ and } |f(x)I - \Delta F_{ex}(I)| < \varepsilon|I|\},$$

From the collection of all δ_ε-fine point-interval pairs $(x, I) \in \Gamma_\varepsilon$, a subset of $[a, b] \in R$ may be obtained, as follows.

Definition 3.1. The set $\{x \in [a, b] \mid \text{for every } \varepsilon > 0 \text{ there exists a } \delta_\varepsilon\text{-fine } (x, I) \in \Gamma_\varepsilon\}$, denoted by $(vp)F[a, b]$, is said to be the null set of F on $[a, b]$.

Definition 3.2. The set $(vs)F[a, b] = [a, b]\backslash(vp)F[a, b]$ is said to be the residual set of F on $[a, b]$.

Clearly, $(vs)F[a, b] = E$. Accordingly, we are in a position to define the notion of a residue of an interval function $\Delta F : I([a, b]) \mapsto R$ at $x \in [a, b]$, [3, 5].

Definition 3.3. An interval function $\Delta F : I([a, b]) \mapsto R$ is said to have a residue at $x \in [a, b]$, with residual value $\mathcal{R}(x)$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon(x) \equiv \underline{\delta_\varepsilon}$ such that $|\Delta F(I) - \mathcal{R}(x)| < \varepsilon$, whenever (x, I) is δ_ε-fine point-interval pair and x is a point of $I \in \mathcal{I}([a, b])$.

A real-valued point function $\mathcal{R} : [a, b] \mapsto R$, which is the limit of ΔF on $[a, b]$, is called a residual function of F on $[a, b]$.

Definition 3.4. For $F : [a, b] \mapsto R$ let $E \subset [a, b]$ be its residual set. The residual function $\mathcal{R}$ of ΔF is said to be basically summable $(BS_{\delta_\varepsilon})$ on E with the sum $\Re \in R$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon(x) \equiv \underline{\delta_\varepsilon}$ such that $|\Sigma_{\Delta F}(P[a, b]|_E) - \Re| < \varepsilon$, whenever $P[a, b]|_E \subset P[a, b]$ and $P[a, b] \in \mathcal{P}[a, b]$ is δ_ε-fine partition. The residual function $\mathcal{R}$ of ΔF is BSG_{δ_ε} on E if E can be written as a countable union of sets on each of which $\mathcal{R}$ is BS_{δ_ε}. In symbols, $\Re = \sum_{x \in E} \mathcal{R}(x)$.

Remark 3.1. If $\Re = 0$, then F has negligible variation on E, [1]. However, if there is a set $E \subset [a, b]$ of variation zero: Given $\varepsilon > 0$, there is a gauge $\delta_\varepsilon(x) \equiv \underline{\delta_\varepsilon}$ such that $|\Sigma_{\Delta x}(P[a, b]|_E)| < \varepsilon$, whenever $P[a, b]|_E \subset P[a, b]$ and $P[a, b] \in \mathcal{P}[a, b]$ is δ_ε-fine partition; on which $\mathcal{R}$ of ΔF is BS_{δ_ε} with $\Re \neq 0$, then F does not satisfy the variational *Strong Lusin* condition on $[a,b]$. On the other hand, since for every $\varepsilon > 0$ there exists a gauge δ_ε such that $|\Delta F(I)| < \varepsilon$, whenever (x,I) is a δ_ε-fine point-interval pair tagged in the null set $(vp)F[a,b]$, and x is a point of $I \in \mathcal{I}([a,b])$, it follows immediately that $\mathcal{R}(x) \equiv 0$ on $(vp)F[a,b]$. In addition, for a given pair of functions F and $\mathcal{R}$, if ΔF is an additive function, and $\mathcal{R}$ vanishes identically on the whole interval $[a,b]$, then $\Delta F([a,b]) = \sum_{x\in[a,b]} \mathcal{R}(x)$. So, if $F_{ex} : [a,b] \mapsto R$ is the primitive of f, defined by (2), then using the *Newton-Leibniz* formula we may obtain that for any compact interval $I \subset [a,b]\backslash E$

$$\sum_{x\in I} \mathcal{R}(x) = \Delta F\,(I) = \int_I f\,(x)\,dx.\blacktriangledown \tag{5}$$

If $E \subset [a,b]$ is any non-empty set of *Lebesgue* measure zero, at whose points any real valued function F can take values $\pm\infty$ or not be defined at all and, in addition, its residual set, then we can divide the infinite sum of all values of the null function $\mathcal{R}$, as a residual function of ΔF on $[a,b]$, into two sums $\sum_{x\in(vp)F[a,b]} \mathcal{R}(x) = vp\int_a^b f\,(x)\,dx$ and $\sum_{x\in E} \mathcal{R}(x)$, so that

$$\Delta F\,([a,b]) = \sum_{x\in[a,b]} \mathcal{R}(x) = vp\int_a^b f\,(x)\,dx + \sum_{x\in E} \mathcal{R}(x). \tag{6}$$

In what follows, we will prove the theorem that gives us this result explicitly. If $vp\int_a^b f\,(x)\,dx$ does not exist, then $vp\int_a^b f\,(x)\,dx + \sum_{x\in E} \mathcal{R}(x)$ is reduced to the so-called indeterminate expression $\infty - \infty$ that actually have, in this situation, the real numerical value of $\Delta F([a,b])$.

Now, we are in a position to define the total value (vt) of the *Riemann* integral of f.

Definition 3.5. For a compact interval $[a,b] \in R$ let $E \subset [a,b]$ be non-empty sets of *Lebesgue* measure zero and $\varphi : \mathcal{I}([a,b]) \mapsto R$ be an interval function whose limit on $[a,b]\backslash E$ is the point function f. The function f is totally *Riemann* integrable to $\Im \in R$ on $[a,b]$, if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon(x) \equiv \underline{\delta}_\varepsilon$ such that $|\Sigma_{\varphi\Delta x}(P[a,b]) - \Im| < \varepsilon$, whenever $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-fine partition. In symbols, $\Im = vt\int_a^b f\,(x)\,dx$.

Remark 3.2. By the previous definition, since $\Sigma_{\varphi\Delta x}(P[a,b]) = \Delta F([a,b])$, where φ is defined by (4), whenever $P[a,b] \in \mathcal{P}[a,b]$, it follows that for any point function f, which has a primitive F defined at the end points of a compact interval $[a,b]$, the generalized *Newton–Leibniz* formula holds $\Delta F([a,b]) = vt\int_a^b f\,(x)\,dx.\blacktriangledown$

Theorem 3.1. *For $[a,b] \in R$ let $E \subset [a,b]$ be non-empty sets of Lebesgue measure zero at whose points a primitive F that is defined and differentiable on $[a,b]\backslash E$ and its derivative f can take values $\pm\infty$ or not be defined at all. If the residual function $\mathcal{R}$ of ΔF is BS_{δ_ε} on E to the sum $\Re \in R$ then f is Riemann integrable* *on* $[a,b]$ *and*

$$\Delta F([a,b]) = vt\int_a^b f(x)\,dx = vp\int_a^b f(x)\,dx + \sum_{x\in E}\mathcal{R}(x). \tag{7}$$

Proof. Let F_{ex} and $D_{ex}F$ be defined by (2). Since the residual function $\mathcal{R}$ of ΔF is BS_{δ_ε} on E to $\Re$, it follows from *Definitions 3.4.* that for every $\varepsilon > 0$ there exist a gauge $\delta_{\varepsilon *}(x) \equiv \underline{\delta}_{\varepsilon *}$ on $[a,b]$ such that $|\Sigma_{\Delta F}(P[a,b]|_E) - \Re| < \varepsilon$, whenever $P[a,b]|_E \subset P[a,b]$ and $P[a,b] \in \mathcal{P}[a,b]$ is $\delta_{\varepsilon *}$-fine partition. In addition, E is the residual set of F on $[a,b]$, and f is the limit of $\varphi_{ex} = \Delta F_{ex}/\Delta x$ on $[a,b]\backslash E$. Therefore, for every $\varepsilon > 0$ there exists a gauge $\delta_{\varepsilon\#}(x) \equiv \underline{\delta}_{\varepsilon\#}$ on $[a,b]$ such that $|(\Sigma_{\Delta F} - \Xi_f)(P[a,b]\backslash P[a,b]|_E)| < (b-a)\varepsilon$, whenever $([a_i,b_i],x_i) \in P[a,b]\backslash P[a,b]|_E$ and $P[a,b] \in \mathcal{P}[a,b]$ is $\delta_{\varepsilon\#}$-fine partition. A gauge $\delta_\varepsilon(x) \equiv \underline{\delta}_\varepsilon$ on $[a,b]$ may be chosen, so that $\delta_\varepsilon = min(\delta_{\varepsilon *}, \delta_{\varepsilon\#})$. Hence, for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon(x) \equiv \underline{\delta}_\varepsilon$ on $[a,b]$ such that

$$|\Xi_{D_{ex}F}(P[a,b]) - [\Delta F([a,b]) - \Re]| =$$

$$= |(\Xi_f - \Sigma_{\Delta F})(P[a,b]\backslash P[a,b]|_E) - \Sigma_{\Delta F}(P[a,b]|_E) - \Re| \le$$

$$\le |(\Xi_f - \Sigma_{\Delta F})(P[a,b]\backslash P[a,b]|_E)| + |\Sigma_{\Delta F}(P[a,b]|_E) - \Re| < [(a-b)+1]\,\varepsilon,$$

whenever $P[a,b] \in \mathcal{P}[a,b]$ is a δ_ε-fine partition.

By *Definition 2.1.*, f is *Riemann* integrable on $[a,b]$ and

$$\Delta F([a,b]) = vt\int_a^b f(x)\,dx = vp\int_a^b f(x)\,dx + \Re.$$

□

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Sistemi inkluzija Hilbertovih modula

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Apstrakt

U ovom radu razmatra se uopštenje pojma sistema inkluzija Hilbertovih prostora. Poznato je da sistem inkluzija Hilbertovih prostora generiše sistem proizvoda Hilbertovih prostora i da postoji izomorfizam između jedinica sistema inkluzija i jedinica generisanog sistema proizvoda.

Ovde se posmatra sistem inkluzija dvostranih Hilbertovih modula nad C^*-algebrom kompaktnih operatora na Hilbertovom prostoru. Dokazuje se da, ukoliko je Hilbertov prostor konačne dimenzije, postoji izomorfizam između jedinica sistema inkluzija i jedinica generisanog sistema proizvoda.

1 Uvod

Sistemi inkluzija Hilbertovih prostora su parametarske familije Hilbertovih prostora, slične sistemima proizvoda, sa razlikom u tome što su unitarna preslikavanja koja povezuju odgovarajuće Hilbertove prostore zamenjena izometrijama. Zapravo, ovi objekti su dosta prisutni u samoj teoriji sistema proizvoda. Pridruživanje sistema proizvoda CP-polugrupama se ostvaruje tako što se prvo formira određeni sistem inkluzija, a potom se, pomoću tehnike induktivnih limesa, dobija sistem proizvoda (detalji se mogu videti u [6]). U [5] su definisani sistemi inkluzija Hilbertovih prostora i upotrebljena je suština pomenutog postupka (iz [6]) da se dokaže da svaki sistem inkluzija indukuje sistem proizvoda delovanjem induktivnih limesa. Takođe je istaknuto da se glavne osobine sistema proizvoda, kao što su npr. postojanje jedinica i struktura morfizama, mogu videti na nivou sistema inkluzija.

O sistemima inkluzija se, otprilike u isto vreme, govori u [12], ali pod nazivom "sistemi potproizvoda". Razmatra se njihova opšta teorija i, takođe, veza sa potpuno pozitivnim (CP) polugrupama. S obzirom na to da postoji opasnost od zabune među terminima "sistemi potproizvoda" i "podsistemi proizvoda", koristimo termin "sistemi inkluzija".

Glavna ideja u ovom radu je da se uopšti pojam sistema inkluzija Hilbertovih prostora iz [5], kao i da se dobiju neki slični rezultati u ovom opštijem slučaju. U tom cilju, posmatramo sisteme inkluzija dvostranih Hilbertovih modula nad C^*-algebrom $\mathcal{B}$, gde je $\mathcal{B}$ C^*-algebra kompaktnih operatora na nekom Hilbertovom prostoru H, tj. $\mathcal{B} = K(H)$.

Poznato je da svaki ograničen $\mathcal{B}$-linearan operator na Hilbertovom $\mathcal{B}$-modulu ima svoj adjungovani operator (sledi iz [9], [7]).

U opštem slučaju, $\mathcal{B}$ nije unitalna C^*-algebra.

U [3], Damir Bakić i Boris Guljaš su opisali Hilbertove C^*-module nad C^*-algebrom (ne obavezno svih) kompaktnih operatora na nekom Hilbertovom prostoru. Ti rezultati su navedeni u Stavu 1 pomenutog rada i u komentarima koji mu prethode. Ovde ih citiramo u Stavu 1.1 i neposredno pre njega.

Neka je E proizvoljan Hilbertov C^*-modul nad C^*-algebrom svih kompaktnih operatora na nekom Hilbertovom prostoru H. Posmatrajmo ideal Hilbert-Šmitovih operatora na H, $\mathcal{C}_2 \subset K(H)$, i neka je

$$E^0_{\mathcal{C}_2} = \mathcal{L}(E\mathcal{C}_2)$$

linearni omotač od $E\mathcal{C}_2$. Očigledno, $E^0_{\mathcal{C}_2}$ je podmodul u E i u isto vreme je (desni) modul nad H^*-algebrom $\mathcal{C}_2$. Skalarni proizvod u E, primenjen na elemente iz $E^0_{\mathcal{C}_2}$, ima vrednost u klasi nuklearnih operatora $\mathcal{C}_1$. Na osnovu toga, $E^0_{\mathcal{C}_2}$ je snabdeveno skalarnim proizvodom $(\cdot,\cdot) = tr(\langle\cdot,\cdot\rangle)$. Označimo normu indukovanu ovim skalarnim proizvodom:

$$\|x\|^2_{\mathcal{C}_2} = tr(\langle x, x\rangle),\ x \in E^0_{\mathcal{C}_2}.$$

Sada je jasno da važi

$$\|x\| \leq \|x\|_{\mathcal{C}_2},\ x \in E^0_{\mathcal{C}_2},$$

gde je $\|\cdot\|$ norma u Hilbertovom C^*-modulu E.

Stav 1.1. Neka je E Hilbertov C^*-modul nad C^*-algebrom $\mathcal{B} = K(H)$, za neki Hilbertov prostor H. Tada postoji Hilbertov H^*-modul

$$E_{\mathcal{C}_2} = \overline{E^0_{\mathcal{C}_2}}^{\mathcal{C}_2} \subset E$$

nad H^*-algebrom $C_2 \subset K(H)$, sa normom $\|x\|^2_{\mathcal{C}_2} = tr(\langle x, x\rangle)$. Za sve $x \in E_{\mathcal{C}_2}$ važi $\|x\| \leq \|x\|_{\mathcal{C}_2}$. Podmodul $E_{\mathcal{C}_2}$ je gust u E u odnosu na normu $\|\cdot\|$ iz E.

Primedba 1.1. Skalarni proizvod $(\cdot,\cdot) = tr(\langle\cdot,\cdot\rangle)$ daje Hilbertovom H^*-modulu $E_{\mathcal{C}_2}$ strukturu Hilbertovog prostora.

2 Sistemi inkluzija i generisani sistemi proizvoda

U ovom poglavlju dajemo definiciju sistema inkluzija dvostranih Hilbertovih $\mathcal{B}-\mathcal{B}$ modula i pokazujemo kako svaki sistem inkluzija generiše odgovarajući sistem proizvoda pomoću induktivnih limesa. Ta tehnika se ne razlikuje bitno od one prikazane u [5].

Definicija 2.1. Hilbertov C^*-modul E nad C^*-algebrom $\mathcal{B}$ je desni $\mathcal{B}$-modul snabdeven skalarnim proizvodom $\langle\cdot,\cdot\rangle : E \times E \to \mathcal{B}$ koji zadovoljava

- $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle\,,\ x, y, z, \in E,\ \alpha, \beta \in \mathbb{C};$
- $\langle x, yb\rangle = \langle x, y\rangle\, b,\ b \in \mathcal{B},\ x, y \in E;$
- $\langle x, x\rangle \geq 0,$
  $\langle x, x\rangle = 0 \Leftrightarrow x = 0,\ x \in E;$
- $\langle x, y\rangle = \langle y, x\rangle^*,$
  odakle sledi $\langle xb, y\rangle = b^* \langle x, y\rangle,\ x, y \in E,\ b \in B;$
- E je kompletan u odnosu na normu $\|\cdot\| = \sqrt{\|\langle\cdot,\cdot\rangle\|}$.

Hilbertov $\mathcal{B}-\mathcal{B}$ modul E je Hilbertov $\mathcal{B}$-modul zajedno sa nedegenerisanom $*$-reprezentacijom C^*-algebre $\mathcal{B}$ pomoću elemenata iz $B^a(E)$, gde je $B^a(E)$ C^*-algebra preslikavanja $E \to E$ koja imaju svoja adjungovana (prema tome su ograničena i desno $\mathcal{B}$-linearna). Homomorfizam $j : \mathcal{B} \to B^a(E)$ koji ostvaruje pomenutu reprezentaciju je kontrakcija, tj. $\|j\| \leq 1$, pa je Hilbertov $\mathcal{B}-\mathcal{B}$ modul E kontraktivan.

Definicija 2.2. Sistem inkluzija (E, β) je familija Hilbertovih $\mathcal{B}-\mathcal{B}$ modula $E = \{E_t, t > 0\}$, zajedno sa familijom dvostranih ($\mathcal{B}-\mathcal{B}$ linearnih) izometrija

$$\beta_{s,t} : E_{s+t} \to E_s \otimes E_t, \quad s, t > 0,$$

gde je $\otimes$ unutrašnji tenzorski proizvod dobijen identifikacijama

$$ub \otimes v \sim u \otimes bv,\ u \otimes vb \sim (u \otimes v)b,\ bu \otimes v \sim b(u \otimes v), \quad u \in E_s, v \in E_t, b \in \mathcal{B},$$

i potom kompletiranjem u odnosu na skalarni proizvod

$$\langle u \otimes v, u_1 \otimes v_1 \rangle = \langle v, \langle u, u_1 \rangle v_1 \rangle, \quad u, u_1 \in E_s,\ v, v_1 \in E_t.$$

Preslikavanja $\beta_{s,t}$ zadovoljavaju

$$(\beta_{r,s} \otimes I_{E_t})\beta_{r+s,t} = (I_{E_r} \otimes \beta_{s,t})\beta_{r,s+t}.$$

(E, β) je sistem proizvoda ako su sva preslikavanja $\beta_{s,t}$ unitarna.

Neka je (E, β) sistem inkluzija. Za $t > 0$, skup J_t je definisan

$$J_t = \{(t_n, t_{n-1}, \dots, t_1) |\ t_i > 0,\ t_1 + \dots + t_n = t,\ n \in \mathbb{N}\}.$$

Za svako $\mathfrak{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$, dužina se definiše kao $|\mathfrak{t}| := t_1 + \dots + t_n = t$.

Za $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$ i $\mathfrak{t} = (t_n, t_{n-1}, \dots t_1) \in J_t$, definiše se zajednički par $\mathfrak{s} \smile \mathfrak{t} \in J_{s+t}$ kao

$$\mathfrak{s} \smile \mathfrak{t} = (s_m, s_{m-1}, \dots, s_1, t_n, t_{n-1}, \dots, t_1) \in J_{s+t}.$$

Na skupu J_t postoji parcijalno uređenje: $\mathfrak{t} \geq \mathfrak{s} = (s_m, s_{m-1}, \dots, s_1)$ ako za svako $i \in \{1, 2, \dots, m\}$ postoji (jedinstveno) $\mathfrak{s}_i \in J_{s_i}$ takvo da je

$$\mathfrak{t} = \mathfrak{s}_m \smile \mathfrak{s}_{m-1} \smile \dots \smile \mathfrak{s}_1.$$

Za $\mathfrak{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$, definiše se

$$E_\mathfrak{t} = E_{t_n} \otimes E_{t_{n-1}} \otimes \dots \otimes E_{t_1}.$$

Za $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \leq \mathfrak{t} = \mathfrak{s}_m \smile \mathfrak{s}_{m-1} \smile \dots \smile \mathfrak{s}_1 \in J_t$, definiše se $\beta_{\mathfrak{t},\mathfrak{s}} : E_\mathfrak{s} \to E_\mathfrak{t}$ na sledeći način:

$$\beta_{\mathfrak{t},\mathfrak{s}} = \beta_{\mathfrak{s}_m, s_m} \otimes \beta_{\mathfrak{s}_{m-1}, s_{m-1}} \otimes \dots \otimes \beta_{\mathfrak{s}_1, s_1}, \tag{1}$$

gde je $\beta_{\mathfrak{s},s} : E_s \to E_\mathfrak{s}$ definisano induktivno kao

$$\beta_{s,s} = I_{E_s},$$

a za $\mathfrak{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$,

$$\beta_{\mathfrak{s},s} = (\beta_{s_m, s_{m-1}} \otimes I)(\beta_{s_m + s_{m-1}, s_{m-2}} \otimes I) \cdots (\beta_{s_m + \dots + s_3, s_2} \otimes I)\beta_{s_m + \dots + s_2, s_1}. \tag{2}$$

Lema 2.1. Za $t > 0$ posmatrajmo parcijalno uređen skup J_t koji je upravo definisan. Familija $(E_{\mathfrak{t}})_{\mathfrak{t}\in J_t}$, zajedno sa familijom preslikavanja $(\beta_{\mathfrak{t},\mathfrak{s}})_{\mathfrak{s}\leq\mathfrak{t}}$, je jedan induktivni sistem dvostranih Hilbertovih $\mathcal{B}-\mathcal{B}$ modula u smislu da važi

1. $\beta_{\mathfrak{s},\mathfrak{s}} = I_{E_{\mathfrak{s}}},\ \mathfrak{s} \in J_t;$
2. $\beta_{\mathfrak{t},\mathfrak{s}}\beta_{\mathfrak{s},\mathfrak{r}} = \beta_{\mathfrak{t},\mathfrak{r}},\ \mathfrak{r} \leq \mathfrak{s} \leq \mathfrak{t} \in J_t.$

Dokaz 2.1. Jedino je 2 potrebno dokazati.
Neka su $\mathfrak{r} = (r_n, \cdots, r_1)$, $\mathfrak{s} = \mathfrak{r}_n \smile \cdots \smile \mathfrak{r}_1$, gde $\mathfrak{r}_i = (r_{ik_i}, \cdots, r_{i1}),\ 1 \leq i \leq n$. Dakle,

$$\mathfrak{t} = (\mathfrak{r}_{nk_n} \smile \cdots \smile \mathfrak{r}_{n1}) \smile (\mathfrak{r}_{(n-1)k_{n-1}} \smile \cdots \smile \mathfrak{r}_{(n-1)1}) \smile \cdots$$
$$\cdots \smile (\mathfrak{r}_{1k_1} \smile \cdots \smile \mathfrak{r}_{11}).$$

Sada je

$$\beta_{\mathfrak{t},\mathfrak{s}}\beta_{\mathfrak{s},\mathfrak{r}} = \beta_{\mathfrak{t},\mathfrak{s}}(\beta_{\mathfrak{r}_n,r_n} \otimes \cdots \otimes \beta_{\mathfrak{r}_1,r_1}) =$$
$$= [\beta_{\mathfrak{r}_{nk_n},r_{nk_n}} \otimes \beta_{\mathfrak{r}_{nk_{n-1}},r_{nk_{n-1}}} \otimes \cdots \otimes \beta_{\mathfrak{r}_{n1},r_{n1}} \otimes \cdots \otimes \beta_{\mathfrak{r}_{1k_1},r_{1k_1}} \otimes \cdots \otimes \beta_{\mathfrak{r}_{11},r_{11}}]$$
$$(\beta_{\mathfrak{r}_n,r_n} \otimes \cdots \otimes \beta_{\mathfrak{r}_1,r_1}) =$$
$$= [\beta_{(\mathfrak{r}_{nk_n}\smile\cdots\smile\mathfrak{r}_{n1}),\mathfrak{r}_n} \otimes \beta_{(\mathfrak{r}_{(n-1)k_{n-1}}\smile\cdots\smile\mathfrak{r}_{(n-1)1}),\mathfrak{r}_{n-1}} \otimes \cdots \otimes \beta_{(\mathfrak{r}_{1k_1}\smile\cdots\smile\mathfrak{r}_{11}),\mathfrak{r}_1}]$$
$$(\beta_{\mathfrak{r}_n,r_n} \otimes \cdots \otimes \beta_{\mathfrak{r}_1,r_1}) =$$
$$= \beta_{(\mathfrak{r}_{nk_n}\smile\cdots\smile\mathfrak{r}_{n1}),r_n} \otimes \beta_{(\mathfrak{r}_{(n-1)k_{n-1}}\smile\cdots\smile\mathfrak{r}_{(n-1)1}),r_{n-1}} \otimes \cdots \otimes \beta_{(\mathfrak{r}_{1k_1}\smile\cdots\smile\mathfrak{r}_{11}),r_1} =$$
$$= \beta_{\mathfrak{t},\mathfrak{r}}.$$

Teorema 2.1. Neka je (E, β) sistem inkluzija (Definicija 2.2), gde je $\mathcal{B}$ C^*-algebra ograničenih linearnih operatora na Hilbertovom prostoru H, tj. $\mathcal{B} = B(H)$, i Hilbertov prostor H je konačne dimenzije. Za svako $t > 0$, neka je

$$\mathcal{E}_t = \operatorname*{indlim}_{\mathfrak{s}\in J_t} E_{\mathfrak{s}}$$

induktivni limes familije $(E_{\mathfrak{s}})_{\mathfrak{s}\in J_t}$ nad J_t. Familija $\mathcal{E} = \{\mathcal{E}_t : t > 0\}$ ima strukturu sistema proizvoda Hilbertovih modula.

Dokaz 2.2. Primetimo da, kako je H konačne dimenzije, $\mathcal{B} = K(H)$ je unitalna C^*-algebra.
Osobine konstrukcije induktivnog limesa su:

1. Postoje kanonske izometrije $i_{\mathfrak{s}} : E_{\mathfrak{s}} \to \mathcal{E}_t$ koje zadovoljavaju $i_{\mathfrak{s}}\beta_{\mathfrak{s},\mathfrak{r}} = i_{\mathfrak{r}}$ za $\mathfrak{r}, \mathfrak{s} \in J_t,\ \mathfrak{r} \leq \mathfrak{s}$.

2. Za $t > 0$, $\mathcal{E}_t = \overline{\operatorname{span}}\{i_{\mathfrak{s}}(a) : a \in E_{\mathfrak{s}},\ \mathfrak{s} \in J_t\}$.

3. (Univerzalno svojstvo) Za dati dvostrani Hilbertov $\mathcal{B}$–$\mathcal{B}$ modul J i izometrije $j_{\mathfrak{s}} : E_{\mathfrak{s}} \to J$ takve da $j_{\mathfrak{s}}\beta_{\mathfrak{s},\mathfrak{r}} = j_{\mathfrak{r}}$ za sve $\mathfrak{r} \leq \mathfrak{s}$, postoji tačno jedna izometrija $j : \mathcal{E}_t \to J$ takva da $j_{\mathfrak{s}} = j i_{\mathfrak{s}}$ za sve $\mathfrak{s} \in J_t$.

4. Neka $K \subset J_t$ ima sledeće svojstvo: Za $\mathfrak{s} \in J_t$ postoji $\mathfrak{t} \in K$ takvo da je $\mathfrak{s} \leq \mathfrak{t}$. Tada skup K nasleđuje uređenje iz J_t i $K \hookrightarrow J_t$ je kofinalna funkcija. Važi $\operatorname{indlim}_{\mathfrak{s}\in J_t} E_{\mathfrak{s}} = \operatorname{indlim}_{\mathfrak{s}\in K} E_{\mathfrak{s}}$.

Na osnovu [6, Stav A.10], $\mathcal{E}_t$ je Hilbertov $\mathcal{B} - \mathcal{B}$ modul. Takođe, prema [6, Primedba A.7], svako $i_{\mathfrak{s}} : E_{\mathfrak{s}} \to \mathcal{E}_t,\ \mathfrak{s} \in J_t$ je dvostrana izometrija.

Za $s, t > 0$ definiše se skup $J_s \smile J_t = \{\mathfrak{s} \smile \mathfrak{t} : \mathfrak{s} \in J_s,\ \mathfrak{t} \in J_t\}$. Za bilo koji element $\mathfrak{r} \in J_{s+t}$, postoje $\mathfrak{s} \in J_s$ i $\mathfrak{t} \in J_t$ takvi da $\mathfrak{s} \smile \mathfrak{t} \geq \mathfrak{r}$. Kako je $J_{\mathfrak{s}} \smile J_{\mathfrak{t}} \subset J_{s+t}$, prema svojstvu 4 konstrukcije induktivnog limesa, važi

$$\mathcal{E}_{s+t} = \operatorname{indlim}_{\mathfrak{r}\in J_{s+t}} E_{\mathfrak{r}} = \operatorname{indlim}_{\mathfrak{s}\smile\mathfrak{t}\in J_s\smile J_t} E_{\mathfrak{s}\smile\mathfrak{t}} = \operatorname{indlim}_{\mathfrak{s}\smile\mathfrak{t}\in J_s\smile J_t} E_{\mathfrak{s}} \otimes E_{\mathfrak{t}}.$$

Za $\mathfrak{s} \in J_s,\ \mathfrak{t} \in J_t$ posmatrajmo preslikavanje $i_{\mathfrak{s}} \otimes i_{\mathfrak{t}} : E_{\mathfrak{s}\smile\mathfrak{t}} \to \mathcal{E}_s \otimes \mathcal{E}_t$, gde su $i_{\mathfrak{s}} : E_{\mathfrak{s}} \to \mathcal{E}_s,\ i_{\mathfrak{t}} : E_{\mathfrak{t}} \to \mathcal{E}_t$ kanonske izometrije. Primetimo da iz

$$\mathfrak{s}' \smile \mathfrak{t}' \leq \mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t \quad \text{sledi} \quad \mathfrak{s}' \leq \mathfrak{s},\ \mathfrak{t}' \leq \mathfrak{t}.$$

Sada, kako je $\beta_{\mathfrak{s}\smile\mathfrak{t},\mathfrak{s}'\smile\mathfrak{t}'} = \beta_{\mathfrak{s},\mathfrak{s}'} \otimes \beta_{\mathfrak{t},\mathfrak{t}'}$, vidimo da važi

$$(i_{\mathfrak{s}} \otimes i_{\mathfrak{t}})\beta_{\mathfrak{s}\smile\mathfrak{t},\mathfrak{s}'\smile\mathfrak{t}'} = i_{\mathfrak{s}}\beta_{\mathfrak{s},\mathfrak{s}'} \otimes i_{\mathfrak{t}}\beta_{\mathfrak{t},\mathfrak{t}'} = i_{\mathfrak{s}'} \otimes i_{\mathfrak{t}'}.$$

Na osnovu univerzalnog svojstva, zaključujemo da postoji jedinstvena izometrija

$$B_{s,t} : \mathcal{E}_{s+t} \to \mathcal{E}_s \otimes \mathcal{E}_t, \quad \text{takva da} \quad B_{s,t} i_{\mathfrak{s}\smile\mathfrak{t}} = i_{\mathfrak{s}} \otimes i_{\mathfrak{t}}. \tag{3}$$

Na osnovu osobine 2 konstrukcije induktivnog limesa, jasno je da je $B_{s,t}$ unitarno preslikavanje.

Na kraju, da bismo dobili još $(B_{r,s} \otimes I_{\mathcal{E}_t})B_{r+s,t} = (I_{\mathcal{E}_r} \otimes B_{s,t})B_{r,s+t}$, dovoljno je da posmatramo vektore oblika $i_{\mathfrak{r}\smile\mathfrak{s}\smile\mathfrak{t}}(x \otimes y \otimes z)$, $x \in E_{\mathfrak{r}},\ y \in E_{\mathfrak{s}},\ z \in E_{\mathfrak{t}}$. Vidimo da važi

$$(B_{r,s} \otimes I_{\mathcal{E}_t})B_{r+s,t} i_{\mathfrak{r}\smile\mathfrak{s}\smile\mathfrak{t}}(x \otimes y \otimes z) = (B_{r,s} \otimes I_{\mathcal{E}_t})(i_{\mathfrak{r}\smile\mathfrak{s}}(x \otimes y) \otimes i_{\mathfrak{t}}(z)) =$$

$$= B_{r,s} i_{\mathfrak{r}\smile\mathfrak{s}}(x \otimes y) \otimes i_{\mathfrak{t}}(z) = i_{\mathfrak{r}}(x) \otimes i_{\mathfrak{s}}(y) \otimes i_{\mathfrak{t}}(z),$$

i, sa druge strane,

$$(I_{\mathcal{E}_r} \otimes B_{s,t}) B_{r,s+t} i_{\mathfrak{r}\smile\mathfrak{s}\smile\mathfrak{t}}(x \otimes y \otimes z) = (I_{\mathcal{E}_r} \otimes B_{s,t})(i_{\mathfrak{r}}(x) \otimes i_{\mathfrak{s}\smile\mathfrak{t}}(y \otimes z)) =$$

$$= i_{\mathfrak{r}}(x) \otimes i_{\mathfrak{s}}(y) \otimes i_{\mathfrak{t}}(z).$$

Definicija 2.3. Sistem proizvoda $(\mathcal{E}, B)$ koji je konstruisan u Teoremi 2.1 se zove sistem proizvoda generisan sistemom inkluzija (E, β).

Napomena 2.1. Ako je (E, β) već sistem proizvoda, onda on generiše sam sebe.

2.1 Jedinice sistema inkluzija

U ovom potpoglavlju govorimo o morfizmima između sistema inkluzija i o jedinicama sistema inkluzija.

Definicija 2.4. Neka su (E, β) i (F, γ) sistemi inkluzija. Neka je $C = (C_t)_{t>0}$ familija dvostranih preslikavanja $C_t : E_t \to F_t$, takva da postoji $p \in \mathbb{R}$ za koje važi $\|C_t\| \leq e^{tp}$, za svako $t > 0$.

C je slab morfizam (ili samo morfizam) ako

$$C_{s+t} = \gamma_{s,t}^* (C_s \otimes C_t) \beta_{s,t}, \quad s, t > 0;$$

C je jak morfizam ako

$$\gamma_{s,t} C_{s+t} = (C_s \otimes C_t) \beta_{s,t}, \quad s, t > 0.$$

Primedba 2.1. Jasno je da je svaki jak morfizam ujedno i slab morfizam, dok obrnuto ne mora da važi. Takođe, ova dva pojma su jednaka kod sistema proizvoda jer su preslikavanja $\beta_{s,t}$ i $\gamma_{s,t}$ unitarna.

Definicija 2.5. Neka je (E, β) sistem inkluzija. Neka je $\xi = (\xi_t)_{t>0}$ familija vektora za koju važi:

1. Za svako $t > 0$, $\xi_t \in E_t$;
2. Postoji $p \in \mathbb{R}$ takvo da je $\|\xi_t\| \leq e^{tp}$, za sve $t > 0$;
3. Za neko $t > 0$, $\xi_t \neq 0$.

Kažemo da je ξ slaba jedinica (ili samo jedinica) ako je

$$\xi_{s+t} = \beta^*_{s,t}(\xi_s \otimes \xi_t), \quad s, t > 0;$$

Kažemo da je ξ jaka jedinica ako je

$$\beta_{s,t}\xi_{s+t} = \xi_s \otimes \xi_t, \quad s, t > 0.$$

Primedba 2.2. Svaka jaka jedinica je ujedno i slaba, dok obrnuto ne mora da važi. Jasno, kod sistema proizvoda, jake i slabe jedinice se poklapaju.

3 Izomorfizam između jedinica sistema inkluzija i jedinica generisanog sistema proizvoda

Neka je C^*-algebra $\mathcal{B} = B(H)$, gde je H Hilbertov prostor konačne dimenzije. (Prema tome, svi operatori u $\mathcal{B}$ su kompaktni.)

Neka je $(\mathcal{E}, B)$ sistem proizvoda generisan sistemom inkluzija (E, β) i neka je $\xi = (\xi_t)$ jedinica u (E, β) za koju postoji neko $p \in \mathbb{R}$ takvo da je za svako $t > 0$ $\|\xi_t\| \leq e^{tp}$.

Neka je $t > 0$. Definišemo

$$E_{\mathfrak{s}} \ni \xi_{\mathfrak{s}} = \xi_{s_m} \otimes \xi_{s_{m-1}} \otimes \cdots \otimes \xi_{s_1}, \quad \mathfrak{s} = (s_m, s_{m-1}, \ldots, s_1) \in J_t.$$

Za sve $\mathfrak{s} \leq \mathfrak{t}$, na osnovu definicije preslikavanja $\beta_{\mathfrak{t},\mathfrak{s}}$ (1), vidimo da je

$$\xi_{\mathfrak{s}} = \beta^*_{\mathfrak{t},\mathfrak{s}}\xi_{\mathfrak{t}}. \tag{4}$$

Prema tome važi:

Lema 3.1. Za $b \in \mathcal{B}$, $(i_{\mathfrak{t}}\xi_{\mathfrak{t}}b)_{\mathfrak{t}\in J_t}$ je konvergentna familija u $\mathcal{E}_t$.

Dokaz 3.1. Za $\mathfrak{t} \geq \mathfrak{s} \in J_t$, važi

$$\langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle - \langle \xi_{\mathfrak{s}}, \xi_{\mathfrak{s}} \rangle = \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle - \left\langle \beta^*_{\mathfrak{t},\mathfrak{s}}\xi_{\mathfrak{t}}, \beta^*_{\mathfrak{t},\mathfrak{s}}\xi_{\mathfrak{t}} \right\rangle = \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle - \left\langle \xi_{\mathfrak{t}}, \beta_{\mathfrak{t},\mathfrak{s}}\beta^*_{\mathfrak{t},\mathfrak{s}}\xi_{\mathfrak{t}} \right\rangle =$$

$$= \left\langle \xi_{\mathfrak{t}}, (I_{E_{\mathfrak{t}}} - \beta_{\mathfrak{t},\mathfrak{s}}\beta^*_{\mathfrak{t},\mathfrak{s}})\xi_{\mathfrak{t}} \right\rangle \geq 0,$$

jer je $\beta_{\mathfrak{t},\mathfrak{s}}\beta^*_{\mathfrak{t},\mathfrak{s}} : E_{\mathfrak{t}} \to E_{\mathfrak{t}}$ projektor ([13, Definicija 1.5.4]).

Dakle, vidimo da je $\langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle_{\mathfrak{t}\in J_t}$ rastuća familija samoadjungovanih operatora u $\mathcal{B}$ koja je uniformno ograničena ($\| \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle \| \leq e^{2tp}$) i, prema tome je jako konvergentna u $\mathcal{B}$.

Za $\mathfrak{t} \geq \mathfrak{s} \in J_t$, koristeći (4) i jednakost $i_{\mathfrak{s}} = i_{\mathfrak{t}}\beta_{\mathfrak{t},\mathfrak{s}}$, dobijamo

$$\|i_{\mathfrak{t}}\xi_{\mathfrak{t}}b - i_{\mathfrak{s}}\xi_{\mathfrak{s}}b\|^2_{\mathcal{E}_t} = \| \langle i_{\mathfrak{t}}\xi_{\mathfrak{t}}b - i_{\mathfrak{s}}\xi_{\mathfrak{s}}b, i_{\mathfrak{t}}\xi_{\mathfrak{t}}b - i_{\mathfrak{s}}\xi_{\mathfrak{s}}b \rangle \|_{\mathcal{B}} =$$

$$= \| \langle i_{\mathfrak{t}}\xi_{\mathfrak{t}}b, i_{\mathfrak{t}}\xi_{\mathfrak{t}}b \rangle - \langle i_{\mathfrak{t}}\xi_{\mathfrak{t}}b, i_{\mathfrak{s}}\xi_{\mathfrak{s}}b \rangle - \langle i_{\mathfrak{s}}\xi_{\mathfrak{s}}b, i_{\mathfrak{t}}\xi_{\mathfrak{t}}b \rangle + \langle i_{\mathfrak{s}}\xi_{\mathfrak{s}}b, i_{\mathfrak{s}}\xi_{\mathfrak{s}}b \rangle \| =$$

$$= \|b^* \langle i_{\mathfrak{t}}\xi_{\mathfrak{t}}, i_{\mathfrak{t}}\xi_{\mathfrak{t}} \rangle b - b^* \langle i_{\mathfrak{t}}\xi_{\mathfrak{t}}, i_{\mathfrak{s}}\xi_{\mathfrak{s}} \rangle b - b^* \langle i_{\mathfrak{s}}\xi_{\mathfrak{s}}, i_{\mathfrak{t}}\xi_{\mathfrak{t}} \rangle b + b^* \langle i_{\mathfrak{s}}\xi_{\mathfrak{s}}, i_{\mathfrak{s}}\xi_{\mathfrak{s}} \rangle b\| =$$

$$= \|b^* \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle b - b^* \langle \xi_{\mathfrak{s}}, \xi_{\mathfrak{s}} \rangle b - b^* \langle \xi_{\mathfrak{s}}, \xi_{\mathfrak{s}} \rangle b + b^* \langle \xi_{\mathfrak{s}}, \xi_{\mathfrak{s}} \rangle b\| =$$

$$= \|b^* \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle b - b^* \langle \xi_{\mathfrak{s}}, \xi_{\mathfrak{s}} \rangle b\|.$$

Kako je $b \in \mathcal{B}$ kompaktan operator i $\langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle_{\mathfrak{t}\in J_t}$ jako konvergira u $\mathcal{B}$, familija $(b^* \langle \xi_{\mathfrak{t}}, \xi_{\mathfrak{t}} \rangle b)_{\mathfrak{t}\in J_t}$ uniformno konvergira u $\mathcal{B}$, pa $(i_{\mathfrak{t}}\xi_{\mathfrak{t}}b)_{\mathfrak{t}\in J_t}$ konvergira u $\mathcal{E}_t$.

Teorema 3.1. Neka je C^*-algebra $\mathcal{B} = B(H)$, gde je H Hilbertov prostor konačne dimenzije. Neka je (E, β) sistem inkluzija i $(\mathcal{E}, B)$ sistem proizvoda njime generisan.

1. Kanonsko preslikavanje $i = (i_t)$, $i_t : E_t \to \mathcal{E}_t$, je izometrički jak morfizam ovih sistema inkluzija.

2. Preslikavanje $i^* = (i^*_t)$ je izomorfizam između jedinica u $(\mathcal{E}, B)$ i jedinica u (E, β).

Dokaz 3.2. 1. Neka je $s, t > 0$. Tada je $(s + t) \leq (s, t) \in J_{s+t}$, pa je

$$i_{(s+t)} = i_{(s,t)}\beta_{(s,t),(s+t)},$$

na osnovu osobine 1 konstrukcije induktivnog limesa (dokaz Teoreme 2.1). Na osnovu (2) je

$$i_{s+t} = i_{(s,t)}\beta_{s,t},$$

pa je

$$B_{s,t}i_{s+t} = B_{s,t}i_{(s,t)}\beta_{s,t} = (i_s \otimes i_t)\beta_{s,t}, \ \forall s,t > 0,$$

prema (3). Dakle, i je jak morfizam.

2. Neka je $\eta = (\eta_t)$ jedinica u $(\mathcal{E}, B)$. Prema tome, postoji $a \in \mathbb{R}$ takvo da $\|\eta_t\| \leq e^{ta}$, za sve $t > 0$. Takođe, $\|i_t^*\eta_t\| \leq e^{ta}$. Sada važi

$$i_{s+t}^*\eta_{s+t} = i_{s+t}^*B_{s,t}^*(\eta_s \otimes \eta_t) = [B_{s,t}i_{s+t}]^*(\eta_s \otimes \eta_t) = [(i_s \otimes i_t)\beta_{s,t}]^*(\eta_s \otimes \eta_t) =$$

$$= (\beta_{s,t}^*(i_s^* \otimes i_t^*))(\eta_s \otimes \eta_t) = \beta_{s,t}^*(i_s^*\eta_s \otimes i_t^*\eta_t),$$

pa je $i^*\eta = (i_t^*\eta_t)$ (slaba) jedinica u (E, β).
Kako je i jak morfizam, odmah sledi da je i^* (slab) morfizam.

Za $\mathfrak{s} = (s_m, s_{m-1}, \ldots, s_1) \in J_t$ definiše se

$$\mathcal{E}_{\mathfrak{s}} = \mathcal{E}_{s_m} \otimes \mathcal{E}_{s_{m-1}} \otimes \cdots \otimes \mathcal{E}_{s_1}.$$

Neka je $i_{\mathfrak{s}} : E_{\mathfrak{s}} \to \mathcal{E}_t$ kanonska izometrija. Slično kao u (2), definišu se preslikavanja $B_{\mathfrak{s},t} : \mathcal{E}_t \to \mathcal{E}_{\mathfrak{s}}$, i korišćenjem (3) dobijamo

$$B_{\mathfrak{s},t}i_{\mathfrak{s}} = i_{s_m} \otimes i_{s_{m-1}} \otimes \cdots \otimes i_{s_1}. \tag{5}$$

Za svaku jedinicu $\eta = (\eta_t)$ u $(\mathcal{E}, B)$ definišemo

$$\eta_{\mathfrak{s}} = \eta_{s_m} \otimes \cdots \otimes \eta_{s_1} \in \mathcal{E}_{\mathfrak{s}}.$$

Tada je

$$B_{\mathfrak{s},t}^*\eta_{\mathfrak{s}} = \eta_t. \tag{6}$$

Injektivnost od i^*:
Neka su η, ζ jedinice u $(\mathcal{E}, B)$ takve da $i_t^*\eta_t = i_t^*\zeta_t$, za sve $t > 0$.

Neka je $t > 0$. Za svako $\mathfrak{s} = (s_m, s_{m-1}, \ldots, s_1) \in J_t$, prema (6) i (5) važi

$$i_{\mathfrak{s}}^*\eta_t = i_{\mathfrak{s}}^*B_{\mathfrak{s},t}^*\eta_{\mathfrak{s}} = (B_{\mathfrak{s},t}i_{\mathfrak{s}})^*\eta_{\mathfrak{s}} = (i_{s_m}^* \otimes i_{s_{m-1}}^* \otimes \cdots \otimes i_{s_1}^*)(\eta_{s_m} \otimes \eta_{s_{m-1}} \otimes \cdots \otimes \eta_1) =$$

$$= i_{s_m}^*\eta_{s_m} \otimes \cdots \otimes i_{s_1}^*\eta_{s_1} = i_{s_m}^*\zeta_{s_m} \otimes \cdots \otimes i_{s_1}^*\zeta_{s_1} =$$

$$= (i_{s_m}^* \otimes i_{s_{m-1}}^* \otimes \cdots \otimes i_{s_1}^*)(\zeta_{s_m} \otimes \zeta_{s_{m-1}} \otimes \cdots \otimes \zeta_{s_1}) =$$

$$= (B_{\mathfrak{s},t}i_{\mathfrak{s}})^*\zeta_{\mathfrak{s}} = i_{\mathfrak{s}}^*B_{\mathfrak{s},t}^*\zeta_{\mathfrak{s}} = i_{\mathfrak{s}}^*\zeta_t,$$

odakle sledi $i_{\mathfrak{s}}i_{\mathfrak{s}}^*\eta_t = i_{\mathfrak{s}}i_{\mathfrak{s}}^*\zeta_t \in \mathcal{E}_t$.

Za $\mathfrak{s} \leq \mathfrak{t} \in J_t$, identitet $i_{\mathfrak{t}}\beta_{\mathfrak{t},\mathfrak{s}} = i_{\mathfrak{s}}$ daje

$$i_{\mathfrak{s}}i_{\mathfrak{s}}^*i_{\mathfrak{t}}i_{\mathfrak{t}}^* = i_{\mathfrak{s}}i_{\mathfrak{s}}^*.$$

Kako je $\mathcal{B} = B(H)$, gde je H konačno dimenzioni Hilbertov prostor, $\mathcal{B}$ zapravo predstavlja C^*-algebru svih kompaktnih operatora na H, odnosno $\mathcal{B} = \mathcal{C}_2$ (Hilbert-Šmitovi operatori na H). Primena Stava 1.1 i Primedbe 1.1, u ovom slučaju, obezbeđuje da je $\mathcal{E}_t = (\mathcal{E}_t)_{\mathcal{C}_2}$ ujedno i Hilbertov prostor sa skalarnim proizvodom $(\cdot,\cdot) = tr(\langle\cdot,\cdot\rangle)$.

Prema tome, kako je $\mathcal{E}_t = \overline{\text{span}}\{i_{\mathfrak{s}}(a) : a \in E_{\mathfrak{s}},\ \mathfrak{s} \in J_t\}$, rastuća familija projektora $(i_{\mathfrak{s}}i_{\mathfrak{s}}^*)$ jako konvergira identičkom operatoru na $\mathcal{E}_t$, pa dobijamo $\eta_t = \zeta_t$.

Surjektivnost od i^*:
Neka je $\xi = (\xi_t)$ (slaba) jedinica u (E, β) za koju je $\|\xi_t\| \leq e^{ta}$ za neko $a \in \mathbb{R}$ i svako $t > 0$.

Neka je $t > 0$. Za svako $\mathfrak{s} = (s_n, \cdots, s_1) \in J_t$, definišimo

$$\xi_{\mathfrak{s}} = \xi_{s_n} \otimes \cdots \otimes \xi_{s_1}.$$

Za $\mathfrak{s} \leq \mathfrak{t} \in J_t$ važi

$$\xi_{\mathfrak{s}} = \beta_{\mathfrak{t},\mathfrak{s}}^*\xi_{\mathfrak{t}}. \tag{7}$$

Na osnovu Leme 3.1, uzimajući $\mathcal{B} \ni b = I$ (identički operator), zaključujemo da familija $(i_{\mathfrak{t}}\xi_{\mathfrak{t}})_{\mathfrak{t}\in J_t}$ konvergira u dvostranom Hilbertovom $\mathcal{B} - \mathcal{B}$ modulu $\mathcal{E}_t$. Označimo njenu graničnu vrednost

$$\lim_{\mathfrak{t}\in J_t} i_{\mathfrak{t}}\xi_{\mathfrak{t}} = \eta_t \in \mathcal{E}_t. \tag{8}$$

Pokazujemo da je $\eta = (\eta_t)$ jedinica u $(\mathcal{E}, B)$.
Za $x_1, x_2, \ldots, x_k \in \mathcal{E}_s$ i $y_1, y_2, \ldots, y_k \in \mathcal{E}_t,\ k \geq 1$,

$$\left\langle B_{s,t}\eta_{s+t}, \sum_l x_l \otimes y_l \right\rangle = \sum_l \left\langle \eta_{s+t}, B_{s,t}^*(x_l \otimes y_l)\right\rangle =$$

$$= \sum_l \lim_{\mathfrak{s}\smile\mathfrak{t}\in J_s\smile J_t} \left\langle i_{\mathfrak{s}\smile\mathfrak{t}}\xi_{\mathfrak{s}\smile\mathfrak{t}}, B_{s,t}^*(x_l \otimes y_l)\right\rangle =$$

$$= \sum_l \lim_{\mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t} \langle (i_{\mathfrak{s}} \otimes i_{\mathfrak{t}})(\xi_{\mathfrak{s}} \otimes \xi_{\mathfrak{t}}), x_l \otimes y_l \rangle =$$

$$= \sum_l \lim_{\mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t} \langle i_{\mathfrak{s}} \xi_{\mathfrak{s}} \otimes i_{\mathfrak{t}} \xi_{\mathfrak{t}}, x_l \otimes y_l \rangle = \sum_l \lim_{\mathfrak{s} \smile \mathfrak{t} \in J_s \smile J_t} \langle i_{\mathfrak{t}} \xi_{\mathfrak{t}}, \langle i_{\mathfrak{s}} \xi_{\mathfrak{s}}, x_l \rangle \, y_l \rangle =$$

$$= \sum_l \langle \eta_t, \langle \eta_s, x_l \rangle \, y_l \rangle = \sum_l \langle \eta_s \otimes \eta_t, x_l \otimes y_l \rangle = \left\langle \eta_s \otimes \eta_t, \sum_l x_l \otimes y_l \right\rangle .$$

Odavde zaključujemo da je η jedinica u $(\mathcal{E}, B)$.

Neka je $x \in E_t$. Koristeći (8), osobinu 1 konstrukcije induktivnog limesa (dokaz Teoreme 2.1), kao i (7), vidimo da važi

$$\langle i_t^* \eta_t, x \rangle = \langle \eta_t, i_t x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_{\mathfrak{r}} \xi_{\mathfrak{r}}, i_t x \rangle = \lim_{\mathfrak{r} \in J_t} \langle i_t^* i_{\mathfrak{r}} \xi_{\mathfrak{r}}, x \rangle =$$

$$= \lim_{\mathfrak{r} \in J_t} \left\langle \beta_{\mathfrak{r},t}^* i_{\mathfrak{r}}^* i_{\mathfrak{r}} \xi_{\mathfrak{r}}, x \right\rangle = \lim_{\mathfrak{r} \in J_t} \left\langle \beta_{\mathfrak{r},t}^* \xi_{\mathfrak{r}}, x \right\rangle = \lim_{\mathfrak{r} \in J_t} \langle \xi_t, x \rangle = \langle \xi_t, x \rangle ,$$

pa dobijamo da je $i_t^* \eta_t = \xi_t$.

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Applications of Lyapunov and T-Lyapunov equations in mechanics

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Abstract

This paper considers Lyapunov and T-Lyapunov matrix equations. Lyapunov equation is a matrix equation of the form $AX + XA^T = E$ which plays a vital role in a number of applications, while T-Lyapunov equation is a matrix equation of the form $AX + X^T A^T = E$. In this paper the relation between these equations will be exploit with purpose of applying obtained results in problems regarding damping optimization in mechanical systems.

1 Introduction

Lyapunov equation is a equation of the form

$$AX + XA^T = E, \tag{1}$$

where $A, E \in \mathbb{R}^{n\times n}$ are given and $X \in \mathbb{R}^{n\times n}$ is unknown matrix. T- Lyapunov equation (sometimes called also Lyapunov equation for T-congruence [1]) is a equation of the form

$$AX + X^T A^T = E, \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$ and $E \in \mathbb{R}^{m\times m}$ are given matrices and the unknown matrix is $X \in \mathbb{R}^{n\times m}$.

Lyapunov equations play a vital role in a number of applications such as matrix eigendecompositions [2], control theory [3], model reduction [4–6], numerical solution of matrix differential Riccati equations [7], image processing [8], eigenvalue assignment problem (EVAP) [9, 10], and many more.

On the other hand, T-Lyapunov equation is still not so known and widely used, although it comes from the theory of Hamiltonian mechanical systems [11].

It can be also used for the singular value decomposition of time varying matrices [12], for $\mathcal{H}^2$ optimal output feedback control for descriptor systems and generalized algebraic Riccati equation for continuous-time descriptor systems [13, 14].

It is well known that Lyapunov equation (1) has a unique solution if and only if $\lambda_i(A)+\lambda_j(A) \neq 0$, for all eigenvalues of A. For small or medium dimensions of matrices the Lyapunov equation can be solved by Bartels-Stewart or Hammarling method (see [15–17]). When E is a symmetric matrix, then the solution X is also symmetric matrix.

T- Lyapunov equation (2) does not have solution if E is not symmetric matrix. This can easily be seen since the left hand side of equation is symmetric, that is $(AX + X^T A^T)^T = AX + X^T A^T$. Further, Lyapunov equation for T-congruence never has a unique solution. If $m = n$, $E = E^T$ and A is regular, then equation (2) has infinitely many solutions of the form $X = A^{-1}(Z + \frac{1}{2}E)$, where Z is arbitrary antisymmetric matrix (i.e. there is $n^2/2 - n$ degrees of freedom, so the dimension of the solution space is $n(n-1)/2$). For the general solution for the general matrix A, see [11].

2 Application of Lyapunov equation in mechanical systems

In this section we will present one approach to a damping optimization of mechanical systems in which Lyapunov equation plays an important role.

For that purpose we first consider a mathematical model of a linear vibrational system:

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{3}$$

where the matrices M, D and K (called mass, damping and stiffness, respectively) are real, symmetric matrices of order n with M and K positive definite. $D = C_u + C_{ext}$ is the damping matrix where C_u represents the internal damping while matrix $C_{ext} = vC_0$ is a positive semidefinite matrix with C_0 which describes the geometry of external damping and v represents viscosity parameter. The internal damping C_u is usually taken to be a small multiple of the critical damping that is,

$$C_u = \alpha C_{crit}, \qquad C_{crit} = M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \tag{4}$$

For example, an n-mass oscillator (Figure 1) can be described by differential equation (3) where M and K are given by

$$M = \operatorname{diag}(m_1, m_2, \ldots, m_n),$$

$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix}$$

where $m_i > 0$, $i = 1, \dots, n$ are masses and $k_i > 0$, $i = 1, \dots, k+1$ are spring constants or stiffnesses.

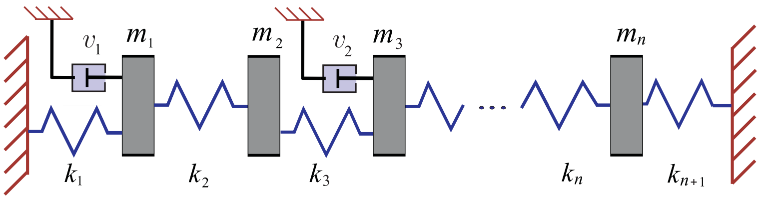

Figure 1: The n-mass oscillator

Damping matrix is of the form $D = C_u + C_{ext}$, where external damping C_{ext} depends on damper's positions and viscosities. For example, oscillator in Figure 1 has two dampers at positions one and three with viscosities v_1 and v_2. This means that $D = C_u + C_{ext} = C_u + v_1 e_1 e_1^T + v_2 e_3 e_3^T$, where e_i is i-th canonical basis vector.

Equation (3) can be transformed to the phase space which yields a system of first order differential equations. Since matrices M and K are positive definite there exists a matrix Φ which simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \dots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I, \tag{5}$$

where $\omega_1^2, \dots, \omega_n^2$ are called undamped eigenfrequencies.

We can write the differential equation (3) in the phase space as

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{6}$$

$$\text{or} \qquad \dot{y} = Ay,$$

where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{7}$$

for more details see [18, 19].

Now, we have the first order differential equation

$$\dot{y} = Ay,$$

with the solution

$$y = e^{At}y_0, \quad \text{where } y_0 \, \text{containes the initial data.}$$

For (M, D, K) from our problem it can be shown that the eigenvalues of A are in the left half of the complex plane, that is, A is stable (see e.g. [19, 20]).

In a damping optimization a very important question arises: for given masses and stiffnesses we want to determine the "best" damping matrix D which insures optimal evanescence of each component of y.

There exists different criterions for that problem, e.g. one criterion is the so-called spectral abscissa criterion, which requires that the maximal real part of the eigenvalues of the corresponding quadratic eigenvalue problem is minimized.

In this paper we would like to use criterion which is based on the minimization of the total energy (as a sum of kinetic and potential energies) averaged over all initial states of the unit total energy and a given frequency range. Moreover, it can be shown that with this averaging our criterion is equivalent to (see e.g. [18])

$$\mathrm{tr}X \quad \to \quad \min, \tag{8}$$

where X is the solution of the Lyapunov equation

$$AX + XA^T = -Z \tag{9}$$

where $Z = GG^T$ and A is as in (7). The case when $G = I$ corresponds to the case when all eigenfrequencies of the undamped system are damped. If we are interested in damping of first s eigenfrequencies of the undamped system that (e.g. if they correspond to the critical part), the matrix G will have the following form

$$G = \begin{bmatrix} I_s & 0 \\ 0 & 0 \\ 0 & I_s \\ 0 & 0 \end{bmatrix}. \tag{10}$$

More details regarding the structure of Z can be found in [18].

Damping optimization using criterion (8) requires solving the Lyapunov equation (9) numerous times since we need to optimize the viscosity parameter or even geometry of external damping. Moreover, efficient damping optimization was widely studied see e.g. [18, 19, 21–24], but due to complexity of presented problem, this is also nowadays very investigated problem.

3 Relations between Lyapunov and T-Lyapunov equations

It is straightforward but very interesting to emphasize that the symmetric solution of T-Lyapunov equation (if it exists) corresponds to the solution of Lyapunov equation, that is Lyapunov equation can be understood as constrained Lyapunov equation for T-congruence:

$$AX + XA^T = E \Longleftrightarrow \begin{array}{r} AX + X^T A^T = E \\ X = X^T \end{array} . \tag{11}$$

The result about existence and uniqueness of the solution of Lyapunov equation can be transferred to T-Lyapunov equation and it reads: Lyapunov equation for T-congruence has unique symmetric solution if and only if $\lambda_i(A) + \lambda_j(A) \neq 0$ for all eigenvalues of A.

Let $A = S\hat{A}S^T$, where S is regular matrix. Then multiplying $AX + XA^T = E$ from the left by S^{-1} and from the right by S^{-T}, from Lyapunov equation we obtain T-Lyapunov equation $\hat{A}Y + Y^T\hat{A}^T = \hat{E}$, where $Y = S^T X S^{-T}$ and $\hat{E} = S^{-1}ES$. Then

$$AX + XA^T = E \Longleftrightarrow \begin{array}{r} \hat{A}Y + Y^T\hat{A}^T = \hat{E} \\ Y = S^T X S^{-T},\ X = X^T \end{array} . \tag{12}$$

If $\hat{A}$ has some particular structure, then the right-hand side constrained structured problem may have better properties then the unstructured Lyapunov equation.

Example 3.1. Using perfect shuffle permutation matrix, Lyapunov equation (9) related to damping optimization can be written in the following form

$$(A_0 - vCC^T)X + X(A_0 - vCC^T)^T = -GG^T. \tag{13}$$

Matrix $A_0 \in \mathbb{R}^{m\times m}$ is block-diagonal matrix ($m = 2n$), while C is a full column rank rectangular matrix with r columns, where usually $r \ll n$. Let A_0 be written as $A_0 = SJS^T$, which can be done in several ways. Then, if we multiply (13) by S^{-1} from the left and by S^{-T} from the right, we obtain T-Lyapunov equation

$$(J - v\hat{C}\hat{C}^T)Y + Y^T(J - v\hat{C}\hat{C}^T)^T = -\hat{G}\hat{G}^T, \tag{14}$$

where $Y = S^T X S^{-T}$, $\hat{C} = S^{-1}C$ and $\hat{G} = S^{-1}G$. The symmetric solution of (14) corresponds to the solution of (13).

One particular choice for S is e.g.

$$S = diag(\omega_1^{1/2}, \omega_1^{1/2}, \omega_2^{1/2}, \omega_2^{1/2}, \ldots, \omega_n^{1/2}, \omega_n^{1/2}) = S^T.$$

Then matrix J is block-diagonal matrix with 2×2 blocks which are all equal to

$$\begin{bmatrix} 0 & 1 \\ -1 & -\alpha \end{bmatrix}.$$

For efficient computation of solutions of (14) which are given by

$$Y = (J - v\hat{C}\hat{C}^T)^{-1}\left(\frac{1}{2}E + Z\right), \quad Z \text{ is antisymmetric}, \tag{15}$$

for different values of v, Sherman-Morrison-Woodbury formula can be used, which gives

$$Y = (J^{-1} + vJ^{-1}\hat{C}(I_r - v\hat{C}^T J^{-1}\hat{C})^{-1}\hat{C}^T J^{-1})\left(\frac{1}{2}E + Z\right). \tag{16}$$

Since J has above structure, J^{-1} is a block-diagonal matrix with blocks

$$\begin{bmatrix} -\alpha & -1 \\ 1 & 0 \end{bmatrix}.$$

Now, the number of flops for calculation of Y as in (16) is $2m^3 + 2rm^2 + 2r^2m + \frac{2}{3}r^3 + O(m^2)$, where $2m^3$ corresponds to matrix product with $\left(\frac{1}{2}E + Z\right)$, $\frac{2}{3}r^3 + 2r^2m$ comes from solving systems with $I_r - v\hat{C}^T J^{-1}\hat{C}$, that is for calculation of matrix $Y_1 := (I_r - v\hat{C}^T J^{-1}\hat{C})^{-1}\hat{C}^T J^{-1}$ and finally term $2rm^2$ corresponds to matrix product $\hat{C}^T J^{-1} Y_1$.

For comparison, direct calculation using formula (15) has complexity $\frac{8}{3}m^3 + 2rm^2 + O(m^2)$, where $\frac{8}{3}m^3$ comes from LU decomposition of matrix $J - v\hat{C}\hat{C}^T$ and solving m systems for columns of Y while $2rm^2$ comes from multiplication $\hat{C}\hat{C}^T$.

On the other hand, solving (13) using standard method has complexity $16m^3$.

In Figure 2 the number of flops for solving T-Lyapunov equation using (16) (denoted by 'Structured T-Lyapunov'), direct calculation using formula (15) (denoted by 'T-Lyapunov') and Bartels-Stewart method for Lyapunov equation (denoted by 'Lyapunov') with respect to dimension m is presented (for $r = 4$). We can see that formula (16) gives the method of choice. In Figure 3 we can see that this conclusion also holds for all $r < \frac{m}{2}$.

4 Conclusions and future work

It this paper it has been pointed out that Lyapunov equation can be concerned as constrained T-Lyapunov equation. One possible application of relation between these two equations has been proposed regarding problems from mechanics.

Equation (14) has infinitely many solutions and finding the symmetric one is not straightforward, but the natural question appears whether the optimization of v (viscosities of dampers) or C_0 (geometry of dampers) can be done efficiently using explicit formula for solutions of (14) and do the solutions of this equation give some additional information about the system.

In the future work we would like to find efficient method for obtaining parameters, i.e. matrix Z which gives symmetric solution of T-Lyapunov equation

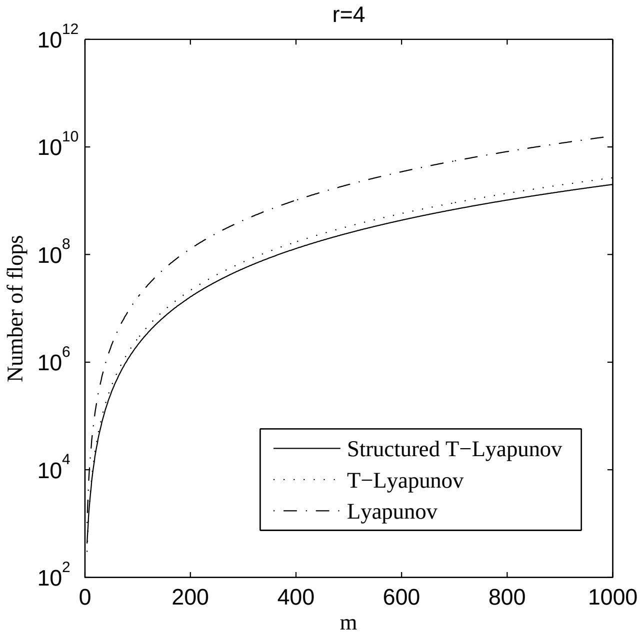

Figure 2: Number of flops with respect to m ($r = 4$)

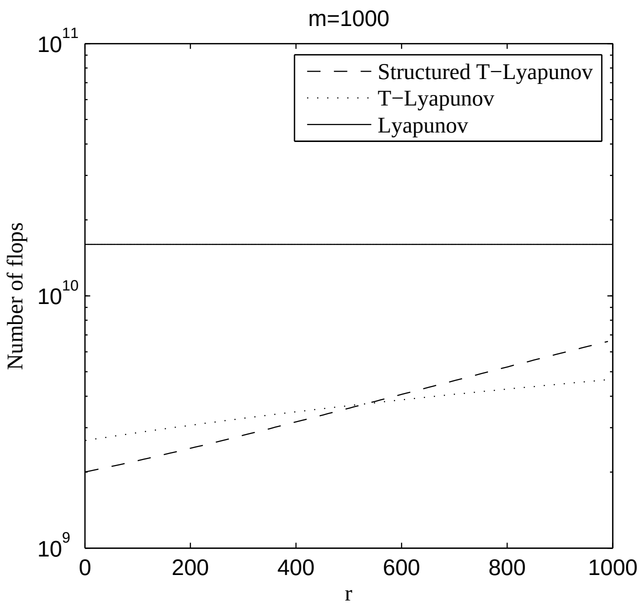

Figure 3: Number of flops with respect to r ($m = 1000$)

(which corresponds to the solution of Lyapunov equation). Also we would like to have better understanding of physical meanings of other solutions of T-Lyapunov equation related to a mechanical systems and then to use this information in order to obtain some results in parameter dependent optimization problems regarding vibrating mechanical systems.

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Hofer's geometry for Lagrangian submanifolds and Hamiltonian diffeomorphisms

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Original scientific paper

Abstract

We give a brief survey of results and problems considering the relation between Hofer's distance for Hamiltonian diffeomorphisms and Lagrangian submanifolds, as well as the role of quasi - autonomous Hamiltonians in the description of geodesics. Besides, we discuss quasi - autonomous Hamiltonians in relation to Hofer's geometry of Hamiltonian diffeomorphisms group within the ambient space of Lagrangian submanifolds.

1 Introduction

Let (P,ω) be a smooth symplectic manifold and $H : P \times [0,1] \to \mathbb{R}$ a smooth (possibly time-dependent) function. Hamiltonian diffemorphism ϕ_t is a solution of the dynamical system

$$\frac{d\phi_t}{dt}(x) = X_H(\phi_t(x)), \quad \phi_0 = \mathrm{Id} \tag{1}$$

where X_H is a Hamiltonian vector field, i.e.

$$\omega(X_H, \cdot) = dH(\cdot). \tag{2}$$

In local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ Hamiltonian diffeomorphism is a solution of the system:

$$\frac{dq_j}{dt} = -\frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = \frac{\partial H}{\partial q_j}.$$

In classical mechanics, these are admissible motions and Hamiltonian H is the energy of the system.

Hamiltonian diffeomorphisms preserve symplectic form, i.e:

$$\phi_t^*\omega = \omega,$$

hence they form a subset of the group of symmetries of symplectic manifold, denoted by $\mathrm{Symp}(P,\omega)$. Moreover, they form a normal subgroup of Symp, i.e:

$$\mathrm{Ham}(P,\omega) \lhd \mathrm{Symp}(P,\omega)$$

with respect to the composition $\circ$. Here $\mathrm{Ham}(P,\omega)$ denotes the group of Hamiltonian diffeomorpshisms.

Let $\varphi(t)$ be any smooth path in $\mathrm{Ham}(P,\omega)$, i.e.

$$\varphi : [0,1] \to \mathrm{Ham}(P,\omega),\ \varphi(0) = \mathrm{Id}\,.$$

A non-trivial fact due to Banyaga [1] is that this is also a family of Hamiltonian diffeomorpshisms, i.e. that the vector field $X := \frac{d\varphi(t)}{dt}$ is Hamiltonian (meaning that $\omega(X,\cdot)$ is an exact form).

2 Hofer's geometry

The pairing $H \leftrightsquigarrow X_H$ is uniquely determined up to a constant (ω is non-degenerate). Therefore, the Lie algebra of $\mathrm{Ham}(P,\omega)$ is naturally identified with $C^\infty(P)/\mathbb{R}$. Any norm $\|\cdot\|$ on $C^\infty(P)/\mathbb{R}$ induces a length on $\mathrm{Ham}(P,\omega)$:

$$lenght\left(\{\phi_t^H\}_{t\in[0,1]}\right) = \int_0^1 \left\|\frac{d\phi_t^H}{dt}\right\| dt = \int_0^1 \|H(\cdot,t)\| dt,$$

and the (pseudo)distance:

$$\rho(\phi,\psi) := \inf\{lenght(\{\phi_t\}) \mid \phi_t \text{ connects } \phi \text{ and } \psi\}.$$

It was proved by Eliashberg and Polterovich in [5] that the choice of L_p norm on C^∞, for $p < \infty$, gives rise to a degenerate pseudo-metric, moreover, if the manifold P is closed, it is identically equal to zero (see also [15]).

However, L_∞ norm on $C^\infty/\mathbb{R}$:

$$\|H(\cdot,t)\|_\infty := \max_{x\in P} H(x,t) - \min_{x\in P} H(x,t) \tag{3}$$

induces a non-degenerate distance ρ on $\mathrm{Ham}(P,\omega)$ which is bi-invariant, meaning that

$$\rho(\varphi\circ\phi,\varphi\circ\psi) = \rho(\phi\circ\varphi,\psi\circ\varphi) = \rho(\phi,\psi),$$

for $\psi,\phi,\varphi \in \mathrm{Ham}(P,\omega)$. This metric is called Hofer's metric. The non-degeneracy of ρ was proven by Hofer [6] for $P=\mathbb{R}^{2n}$ and by Polterovich [14], Lalonde and McDuff [7] in the general case.

3 Generalization: Lagrangian submanifolds

An embedding $L \overset{\imath}{\hookrightarrow} P$ into a symplectic manifold (P,ω) is said to be *Lagrangian* if $\dim L = \frac{1}{2}\dim P$ and $\imath^*\omega = 0$.

Remark 3.1. Lagrangian submanifolds are generalization of Hamiltonian diffeomorphisms since the graph

$$\mathrm{Graph}(\phi) := \{(x,\phi(x) \mid x \in P\} \subset P\times P$$

is Lagrangian submanifold of a symplectic manifold $(P\times P, \omega\oplus -\omega)$.

Let L and L_0 be two Lagrangian submanifolds of P. We say that L is *Hamiltonian isotopic* to L_0 if there exists $\phi\in\mathrm{Ham}(P,\omega)$ s.t. $L=\phi(L_0)$. We denote by $\mathcal{L}(L_0,P,\omega)$ the set of all Lagrangian submanifolds that are Hamiltonian isotopic to L_0.

In view of the Remark 3.1 we can say that

$$\mathrm{Ham}(P,\omega) \hookrightarrow \mathcal{L}(\Delta, P\times P, \omega\oplus-\omega), \quad \phi\mapsto \mathrm{Graph}(\phi).$$

3.1 Hofer's metric on $\mathcal{L}(L_0,P,\omega)$

It is possible to define the Hofer distance on the space $\mathcal{L}(L_0,P,\omega)$. The length of the path is defined as

$$\mathit{length}(\{L_t\}_{t\in[0,1]}) := \inf\left\{\int_0^1 \|H_t\|_\infty dt \mid H_t \rightsquigarrow \phi_t^H,\ \phi_t(L_0)=L_t\right\},$$

and the distance between two Lagrangian submanifolds as

$$d(L,L') := \inf\{\mathit{length}(\{L_t\}) \mid L_t \text{ connects } L \text{ and } L'\}.$$

Chekanov [3] proved that, for closed L and tame P (which means that it is somehow geometrically bounded) the above metric is actually non-degenerate.

It is obvious that $length(\{\phi_t^H(L_0)\}) \leq lenght(\{\phi_t^H\})$, so $d(L, \phi(L)) \leq \rho(\mathrm{Id}, \phi)$ and $d(\Delta, \mathrm{Graph}(\phi)) \leq \rho(\mathrm{Id}, \phi)$. The natural question that arises here is weather $\mathrm{Ham}(P, \omega) \hookrightarrow \mathcal{L}(\Delta, P \times P, \omega \otimes -\omega)$ is an isometry (i.e. weather $\rho = d$)? The answer is negative in general and it was given by Ostrover [12]. He proved that if P is closed with $\pi_2(P) = 0$, then there exists a family $\phi_s \in \mathrm{Ham}(P, \omega)$ such that

- $\rho(\mathrm{Id}, \phi_s) \overset{s\to\infty}{\longrightarrow} \infty$
- $d(\mathrm{Graph}(\mathrm{Id}), \mathrm{Graph}(\phi_s)) = const > 0.$

4 Geodesics and quasi-autonomous Hamiltonians

Quasi-autonomous Hamiltonians are generalization of autonomous Hamiltonians.

Definition 4.1. $H : P \times [0,1] \to \mathbb{R}$ is **quasi-autonomous** if there exists $x_+, x_- \in M$ s.t.

$$\max_x H(t,x) = H(t, x_+), \quad \min_x H(t,x) = H(t, x_-), \text{ for all } t \in [0,1].$$

It turns out that quasi-autonomous Hamiltonians play an important role in the description of geodesics of Hofer's metric. Here by geodesic we mean the path that locally minimizes the distance. More precisely, the path $\phi_t \in \mathrm{Ham}(P, \omega)$ is geodesic if for any $t_0 \in [0,1]$ there exists a $\delta > 0$ such that for all $t_1, t_2 \in (t_0 - \delta, t_0 + \delta)$ it holds

$$lenght\left(\{\phi_t\}_{t\in[t_1,t_2]}\right) = \rho(\phi_{t_1}, \phi_{t_2}).$$

The following theorem was proved by Bialy and Polterovich [2], for $\mathbb{R}^{2n}$, and by Lalonde and McDuff [7] for general case.

Theorem 4.1. *The path ϕ_t^H is geodesic in $Ham(P, \omega)$ if and only if H is quasi-autonomous.*

There is an analogue of the previous theorem which describe geodesics in the space $\mathcal{L}(P, L, \omega)$, in the case of the cotangent bundle $P = T^*M$ with the standard symplectic form, $\omega = -d\lambda$ and the zero section $L_0 = O_M$.

Theorem 4.2. *[8] The path $\{L_t\}$ is geodesic in $\mathcal{L}(T^*M, O_M, -d\lambda)$ if and only if it is generated by a quasi-autonomous Hamiltonian.*

5 Spectral invariants, admissible Hamiltonians and $L-$quasi autonomous Hamiltonians

In this section we compare Hofer's length of the path of Lagrangian submanifolds with certain numerical invariants associated to a Hamiltonian functions, called *spectral* or *symplectic* invariants.

More precisely, let (P, ω) be closed symplectic manifold and $L \subset P$ a closed Lagrangian submanifold. Suppose that the following topological property is fulfilled:

$$\pi_2(P, L) = 0. \tag{4}$$

For given Hamiltonian H and a singular homological class $a \in H_*^{\mathrm{sing}}(L)$ there exists a real number $c(a, H)$ with the following properties:

(A) $c(a, H) \in \mathrm{Spec}(\mathcal{A}_H)$, where $\mathrm{Spec}(\mathcal{A}_H)$ is the set of critical values of action functional:

$$\mathcal{A}_H(\alpha) := \int_{\tilde{\alpha}} \omega - \int_{\alpha} H dt.$$

Here α is a smooth path

$$\alpha : [0, 1] \to P, \ \alpha(0), \alpha(1) \in L$$

and $\tilde{\alpha}$ is any smooth mapping from the half-disc $D^+ = \{x^2 + y^2 \leq 1, y \geq 0\}$ to P, such that $\tilde{\alpha}|_{\mathbb{S}^+} = \alpha$, where $\mathbb{S}^+$ is the upper half-circle (here we suppose that $[\alpha] = 0 \in \pi_1(P, L)$). Action functional is well defined due to the condition (4).

(B) $c(a, \cdot)$ is continuous with respect to Hofer distance, more precisely, it holds:

$$|c(a, H_1) - c(a, H_2)| \leq \|H_1 - H_2\|_\infty,$$

where $\|\cdot\|_\infty$ is a Hofer distance.

(C) (Poincaré duality) if we denote by $\widetilde{H}(x, t) := -H(x, 1 - t)$, then

$$c(1, \widetilde{H}) = -c([L], H).$$

(D) (triangle inequality) $c(a \cup b, H \sharp K) \leq c(a, H) + c(b, K)$, where

$$H \sharp K(x, t) = H(x, t) + K((\phi_t)^{-1}(x), t).$$

Here ϕ_t is a Hamiltonian diffeomorphism generated by H in the sense of the equations (1) and (2).

Spectral invariants in Floer theories (in Hamiltonian or Lagrangian case) were studied by Viterbo [17], Oh [9–11], Schwarz [16], etc. They are deeply connected with Hofer's distance. For example, one can use spectral invariants to derive the non-degeneracy of Hofer's distance, which is highly non-trivial fact.

One construction of spectral invariants relies on certain isomorphisms between Morse homology (which is isomorphic to the singular homology) and Floer homology. These are so-called PSS isomorphisms and they were originally constructed by Piunikhin-Salamon-Schwarz in [13]. There are a lot of auxiliary structures and notions involved in the construction of PSS isomorphisms and spectral invariants, but they turn out to be independent of them all. The precise construction of spectral invariants in our particular case can be found in [4].

In this note we establish another connection between spectral invariants and Hofer's length. Similar result was made by Schwarz in [16] in the case of periodic orbit spectral invariants. First we need to introduce some notions.

Definition 5.1. Relative spectral invariant is by definition

$$\gamma(a, H) := c(a, H) - c(1, H),$$

and particularly

$$\gamma(H) := \gamma([L], H).$$

Definition 5.2. We say that H is **admissible** if all the solutions of

$$x \in C^\infty([0,1], P) \mid \dot{x}(t) = X_H(x(t)), x(0), x(1) \in L, [x] = 0 \in \pi_1(P, L)$$

are necessarily constant.

Definition 5.3. . $H : P \times [0,1] \to \mathbb{R}$ is $L-$**quasi-autonomous** if there exists two different points $x_\pm \in L$ and two disjoint neighbourhoods $U_\pm \ni x_\pm$ in P, such that

$$\max_{x\in P} H(x,t) = H(y,t), \quad \text{for all } x \in U_+,\ t \in [0,1],$$
$$\min_{x\in P} H(x,t) = H(y,t), \quad \text{for all } x \in U_-,\ t \in [0,1].$$

The main result is the following.

Theorem 5.1. *Let H be $L-$quasi-autonomous Hamiltonian which is admissible and homotopic to* 0 *through admissible Hamiltonians. Then it holds*

$$\gamma(H) = \|H\|_\infty,$$

where $\|\cdot\|_\infty$ is the Hofer norm (3).

The details of the proof of Theorem 5.1 will be given elsewhere.

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ČETVRTA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
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O razlaganju linearno uredjenih struktura

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Apstrakt

Jedan od alata modelsko teoretske analiza linearnih uredjenja je kondenzacija. Daćemo pregled poznatih kondenzacija i njihovih iteracija, i definisaćemo novu, diskretnu kondenzaciju, odredjenu definabilnom relacijom $\sim$. Neka je $\mathcal{A} = (A, <)$ linearno uredjenje. Definisšimo:

$a \sim b$ ako i samo ako je interval $[a, b]$ diskretno linearno uredjen.

Relacija $\sim$ je definabilna relacija ekvivalencije, a njene klase su konveksni skupovi. Definabilnost ove kondenzacije je ključna i pomoću nje ćemo pokazati: svako linearno uredjenje sa prebrojivo mnogo disjunktnih unarnih predikata je interpretabilno u čistom linearnom uredjenju.

1 Uvod

Ovaj deo rada je pregled oblasti u kojoj se kondenzacije koriste za analizu lineranih uredjenja. Detalji se mogu naći u [1].

Neka su $\mathcal{A} = (A, <_A)$ i $\mathcal{B} = (B, <_B)$ dva linearna uredjenja. Bez umanjenja opštosti, neka su A i B disjunktni (ukoliko nisu, možemo A zameniti sa $A \times \{0\}$ i B sa $B \times \{1\}$). Sa $\mathcal{A} + \mathcal{B}$ ćemo obeležavati linearno uredjenje koje nastaje kad na $\mathcal{A}$ nadovežemo $\mathcal{B}$. Detaljnije, $\mathcal{A} + \mathcal{B}$ je struktura $(A \cup B, <)$ gde je relacija $<$ odredjena sa:
- ako su $a_1, a_2 \in A$, onda $a_1 < a_2$ ako i samo ako je $a_1 <_A a_2$,
- ako su $b_1, b_2 \in B$, onda $b_1 < b_2$ ako i samo ako je $b_1 <_B b_2$ i
- $a < b$ kad god je $a \in A$ i $b \in B$.

Pojam nadovezivanja linearnih uredjenja se lako može uopštiti. Neka je $\mathbf{L} = (L, <_L)$ linearno uredjenje i neka je $\mathcal{A}_l = (A_l, <_l)$ linearno uredjenje za svako $l \in L$. Bez umanjenja opštosti, neka su A_{l_1} i A_{l_2} disjunktni kad god su l_1 i l_2 različiti. Neka je $A = \bigcup_{l \in L} A_l$ i relacija $<$ definisana na A na sledeći način:

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- ako su $a_1, a_2 \in A_l$, onda $a_1 < a_2$ ako i samo ako je $a_1 <_l a_2$;
- ako je $a_1 \in A_{l_1}$ i $a_2 \in A_{l_2}$, onda $a_1 < a_2$ ako i samo ako je $l_1 <_L l_2$.

Očigledno je $(A, <)$ linearno uredjenje. Obeležavaćemo ga sa $\sum_{l \in L} \mathcal{A}_l$ i reći ćemo da je suma linearnih uredjenja $\{\mathcal{A}_l \mid l \in L\}$.

Uredjenje $\sum_{l \in L} \mathcal{A}$ ćemo češ će obeležavati sa $\mathcal{A} \times \mathbf{L}$.

Definicija 1.1. Linearno uredjenje je:

D : diskretno ako svaki element, sem eventualno najvećeg ima neposrednog prethodnika i svaki element, sem eventualno najmanjeg ima neposrednog sledbenika.

R : rasuto ako mu $(\mathbb{Q}, <)$ nije poduredjenje.

G : gusto ako izmedju svaka dva elementa postoji bar još jedan.

Primer 1.1. (1) Jednočlano uredjenje $\mathbf{1}$ je i diskretno i rasuto i gusto uredjenje.

(2) Strukture $\mathbf{n}$, $(\omega, <)$, $(\omega^*, <)$, $(\omega + \omega^*, <)$, $(\mathbb{Z} \times \mathbb{Z}, <)$, $(\mathbb{Z} \times \omega, <)$ i $(\omega + \mathbb{Z} \times \mathbb{Z} + \omega^*, <)$ su i diskretna i rasuta uredjenja.

(3) Strukture $(\omega + \mathbb{Z} \times \mathbb{Q}, <)$, $(\mathbb{Z} \times \mathbb{Q} + \omega^*, <)$, $(\omega + \mathbb{Z} \times \mathbb{Q} + \omega^*, <)$ su diskretna ali nisu rasuta uredjenja

(4) Strukture $(\omega \times \mathbb{Z}, <)$, $(\omega^* \times \mathbb{Z}, <)$, $(\omega^* \times \omega, <)$, $(\omega \times \omega^*, <)$, $(\omega \times \mathbf{n}, <)$, $(\omega^n, <)$ i $(\omega^\omega, <)$ su primeri rasutih uredjenja koja nisu diskretna.

Upravo postojanje ili nepostojanje najmanjeg ili navećeg elementa govori da teorija diskretnih linearnih uredjenja nije kompletna. Postoji pet varijanti mogućih kompletiranja, s tim što peta varijanta sadrži prebrojivo mnogo aksioma, dok se ostale mogu aksiomatizovati konačnim skupom aksioma. Neka su Φ_i sledeće rečenice:

Φ_0: Relacija $<$ linearno uredjuje domen.

Φ_1: Svaki element, sem eventualno najvećeg ima neposrednog prethodnika i svaki element, sem eventualno najmanjeg ima neposrednog sledbenika.

Φ_2: Postoji namanji element.

Φ_2': Ne postoji namanji element.

Φ_3: Postoji najveći element.

Φ_3': Ne postoji najveći element.

Φ_{3+n}: Postoji tačno n elementa.

Φ_{3+n}': Negacija aksiome Φ_{3+n}.

Aksiome Φ_0 i Φ_1 čine teoriju diskretnog linearnog uredjenja.

Neka je $T_1 = \{\Phi_0, \Phi_1, \Phi_2, \Phi_3'\}$. To je potpuna teorija koja kaže da su njeni modeli diskretna uredjenja koja imaju najmanji i nemaju najveći element. Svi njeni modeli su oblika $\mathcal{M}_1 = (\omega + \mathbb{Z} \times L, <)$ za L bilo kakvo linearno uredjenje ili $L = \emptyset$.

Teorija $T_2 = \{\Phi_0, \Phi_1, \Phi_2', \Phi_3\}$ je drugo kompletiranje teorije diskretnih linearnih uredjenja. Njeni modeli nemaju najmanji i imaju najveći element. To su do na izomorfizam $\mathcal{M}_2 = (\mathbb{Z} \times L + \omega^*, <)$ za L bilo kakvo linearno uredjenje ili $L = \emptyset$.

Teorija $T_3 = \{\Phi_0, \Phi_1, \Phi_2', \Phi_3'\}$ je još jedno kompletiranje teorije diskretnih linearnih uredjenja čiji modeli nemaju ni najmanji ni najveći element: To su do na izomorfizam modeli oblika $\mathcal{M}_3 = (\mathbb{Z} \times L, <)$ za L bilo kakvo linearno uredjenje ili $L = \emptyset$.

Neka je $T_{3+n} = \{\Phi_0, \Phi_1, \Phi_{3+n}\}$ za neko $n > 1$. To je kompletna teorija i svi njeni modeli su konačna linearna uredjenja sa tačno n elemenata. Očigledno da su Φ_2 i Φ_3 (postojanje najmanjeg i najvećeg elemnta) posledice teorija T_{3+n}.

Teorija $T_0 = \{\Phi_0, \Phi_1, \Phi_2, \Phi_3\} \cup \{\Phi_{3+n}' | n > 1\}$ je poslednje kompletiranje teorije diskretnih linearnih uredjenja. Njeni modeli su beskonačni i imaju i minimum i maksimum. To su do na izomorfizam modeli $\mathcal{M}_0 = (\omega + \mathbb{Z} \times L + \omega^*, <)$ za L bilo kakvo linearno uredjenje ili $L = \emptyset$.

Više o ovome čitalac može naći u [2].

Svako od linearnih uredjenja koja su rasuta a nisu diskretna datih u primeru 1.1 nisu diskretna jer imaju ili element koji nema neposrednog prethodnika a on sam nije minimalan ili element koji nema neposrednog sledbenika a nije maksimalan. Na primer, ako $(\omega \times \mathbf{n}, <)$ vidimo kao n puta nadovezanu kopiju ω-e, onda su ti elementi redom sve kopije nule sem prve.

Definicija 1.2. Za element linearnog uredjenja koji nema neposrednog prethodnika kažemo da je levi granični.
Za element linearnog uredjenja koji nema neposrednog sledbenika kažemo da je desni granični.
Element je granični, ako je levi ili desni granični.
Dva elementa su susedna ako izmedju njih nema drugih elemenata. Za broj elemenata susednih elementu a koristićemo oznaku $N(a)$.

Napomena 1.1. Važe sledeće:

- Svaki element može najviše imati dva susedna elementa;
- Element je granični akko ima manje od dva susedna elementa;
- Uredjenje je gusto akko je $N(a) = 0$ za svaki njegov element a;
- Uredjenje koje ima više od jednog elementa je diskretno akko
  1. Ima najviše dva granična elementa;

2. Ako je a granični, onda je $N(a) = 1$;
3. Sa tačno jedne strane graničnog nema drugih elemenata domena.

- Rasuto uredjenje nije diskretno akko ima granični element a takav da je ili $N(a) = 0$ ili postoje elementi domena sa obe strane elementa a.

2 Pojam i primeri kondenzacija

U ovom poglavlju ćemo uvesti dva pojma: pojam kondenzacije kao relacije ekvivalencije i pojam kondenzacije kao preslikavanja

Neka je $\mathcal{A} = (A, <)$ linearno uredjenje i neka su I_1 i I_2 dva konveksna disjunktna podskupa skupa A. Ako za jedan, proizvoljno izabran, element skupa I_1 (označimo ga sa a_1) i jedan proizvoljno izabran element skupa I_2 (označimo ga sa a_2) važi $a_1 < a_2$, onda će svi elementi skupa I_1 biti manji od svih elemenata skupa I_2. Na taj način se uredjenje skupa A prirodno produžava na skup disjunktnih konveksnih podskupova skupa A:

$$I_1 < I_2 \text{ akko } a_1 < a_2 \text{ za sve } a_1 \in I_1 \text{ i sve } a_2 \in I_2.$$

A_1 je konveksno razlaganje skupa A ako je kolekcija medjusobno disjunktnih konveksnih podskupova čija unija je ceo A. Očigledno je da relacija $<$, nasledjenja iz sturkture $(A, <)$, linearno uredjuje A_1. Tada ćemo i za strukturu $\mathcal{A}_1 = (A_1, <)$ reći da je konveksno razlaganje strukture $\mathcal{A} = (A, <)$. Svakom razlaganju $\mathcal{A}_1$ odgovara relacija ekvivalencije definisana sa:

$$a \sim b \text{ akko } a \text{ i } b \text{ pripadaju istom elementu skupa } A_1.$$

Njene klase ekvivalencije su baš elementi strukture $\mathcal{A}_1$. Dakle, $\mathcal{A}_1 \cong (A/_\sim, <)$ što znači da se zadavanjem konveksnog razlaganja, definiše i odgovarajuća ekvivalencija. Važi i obrnuto, zadavanjem relacije ekvivalencije čije su klase konveksne, definiše se i razlaganje.

Preslikavanje $f : (A, <) \longrightarrow (B, <)$ je homomorfizam ako iz $a_1 \leqslant a_2$ sledi $f(a_1) \leqslant f(a_2)$. Svaki homomorfizam definiše jednu relaciju ekvivalencije sa konveksnim klasama: $a \sim b$ akko $f(a) = f(b)$. Tada je $(f(A), <) \cong (A/_\sim, <)$.

Homomorfizam $f : (A, <) \longrightarrow (A, <)$ slika klasu relacije $\sim$ u izabranog predstavnika.

Obrnuto, svaka relacija ekvivalencije sa konveksnim klasama definiše homomorfizam kojim se element slika u svoju klasu.

Definicija 2.1. Kondenzacija je svaki epimorfizam linearnih uredjenja t.j. kondenzacija je surjektivno $f : (A, <) \longrightarrow (B, <)$ za koje važi

$$a_1 \leqslant a_2 \text{ akko } f(a_1) \leqslant f(a_2).$$

U tom slučaju kažemo da se $(A, <)$ kondenzovalo u $(B, <)$. Takodje, za relaciju ekvivalencije $\sim$ definisanu na $(A, <)$, kažemo da je kondenzacija ako su joj klase konveksni skupovi u $(A, <)$.

Ova dvostruka definicija kondenzacije, ne bi trebalo da pravi zabunu jer postoji uzajamno jednoznačna korespodencija koja svakoj kondenzaciji definisanoj kao preslikavanju, dodeljuje tačno jednu kondenzaciju definisanu kao relaciju ekvivalencije i obrnuto.

Najpoznatiji primer je kondenzacija čije su klase rasuta uredjenja.

Primer 2.1. Neka je $\mathcal{A}$ bilo koje uredjenje i $x \sim y$ ako i samo ako je interval sa krajevima x i y rasuto uredjenje. Tada je $c_R : \mathcal{A} \longrightarrow \mathcal{A}/_\sim$ definisano sa $c_R(x) = x/_\sim$ jedna kondezacija i $\mathcal{A}/_\sim$ je gusto uredjenje.

Dakle, $\mathcal{A}$ se dobija kad umesto svake tačke gustog uredjenja $\mathcal{A}/_\sim$ stavimo rasuto uredjenje. Posledica ovog razmatranja je

Hauzdorfova teorema:

Svako linearno uredjenje je gusta suma rasutih uredjenja.

3 Iteracije kondenzacija

U naxoj definiciji kondenzacije smo krenuli od konkretnog uredjenja i to uredjenje kondenzovali. Od interesa su kondenzacije definisane na nekoj klasi linearnih uredjenja. Ako je π kondenzacija definisana za sva uredjenja i njome se $\mathcal{A} = (A, <)$ kondenzovalo u $\pi(\mathcal{A}) \cong \mathcal{A}/_\sim$, onda se možemo pitati u šta će se kondenzovati $\pi(\mathcal{A})$ primenom iste kondenzacije. U tom slučaju je prirodno $\pi(\pi(\mathcal{A}))$, označiti sa $\pi^2(\mathcal{A})$ i posmatrati π^2 kao zasebnu kondenzaciju. Naravno, postupak se može dalje nastaviti i tad govorimo o iteraciji kondenzacija. Za razumevanje iteriranih kondezacija, pogodno je imati sliku kondenzacije kao preslikavanja koje element slika u konveksan skup kome sam taj element pripada. Tada će iterirane kondenzacije širiti konveksan skup u koji slikamo element, tj. širiti klasu elementa.

Definicija 3.1. Neka je π kondenzacija. Tada

- $\pi^0(x) = x$;
- $\pi^{\beta+1}(x) = \{y \mid \pi(\pi^\beta(x) = \pi(\pi^\beta(y)\}$;
- $\pi^\alpha(x) = \bigcup\{\pi^\beta(x) \mid \beta < \alpha\}$.

Primer 3.1. Konačna kondenzacija:

$$\pi_K(x) = \{y| \text{ interval s krajevima } x \text{ i } y \text{ je konačan}\}.$$

Konačna kondenzacija ostavlja gusta uredjenja nepromenjena.

$\pi_K^\alpha(\mathbf{1}) \cong \mathbf{1}$ i $\pi_K^\alpha(\mathbb{Q}) \cong \mathbb{Q}$, za svaki ordinal α.

Kako su u ω svaka dva elementa na konačnom rastojanju, biće $\pi_K(\omega) \cong \mathbf{1}$.
$\pi_K^2(\omega) \cong \pi_K(\mathbf{1}) \cong \mathbf{1}$. Za $\alpha > 1$ je $\pi_K^\alpha(\omega) \cong \mathbf{1}$
Svaki element uredjenja ω^n je na konačnom rastojanju od graničnog, pa se klase mogu identifikovati sa graničnim elementima. Zato je
$\pi_K(\omega^n) \cong \omega^{n-1}$

Slično tome se svaka klasa iz π^2 može identifikovati sa graničnim elementima graničnih elemenata, pa je
$\pi_K^2(\omega^n) \cong \omega^{n-2}$
$\vdots$
$\pi_K^n(\omega^n) \cong \mathbf{1}$
Za $\alpha > n$ je $\pi_K^\alpha(\omega^n) \cong \mathbf{1}$

Pažnja: $\pi_K^n(\omega^\omega) \cong \omega^\omega$, za svako n, ali $\pi_K^\omega(\omega^\omega) \cong 1$.
Razlog za to je xto je za element $x \in \omega^\omega$ postoji $n < \omega$ takav da je $\pi^n(x) = \pi^n(0)$, pa će se u ω iteracija svi slikati u jednočlano uredjenje.

činjenica da je $\pi_K^{n-1}(\omega^n) \ncong \mathbf{1}$ i da je $\pi_K^n(\omega^n) \cong \mathbf{1} \cong \pi_K^\alpha(\omega^n)$ za sve ordinale $\alpha > n$ sugerixe da se konačnom kondenzacijom može u izvesnom smislu analizitari složenost linearnog uredjenja tako xto se uvede pojam ranga linearnog preslikavanja. Za to je potrebna sledeća lema koja se može naći u [1].

Lema 3.1. Za svako uredjenje $\mathcal{A}$ i svaku kondenzaciju π, postoji ordinal α takav da je $\pi^\alpha(x) = \pi^\beta(x)$, za sve $x \in A$ i sve ordinale $\beta > \alpha$.

Dokaz: Pretpostavimo da $\pi^\alpha(x) \neq \pi^{\alpha+1}(x)$ važi za sve ordinale. Tada je $\pi^\alpha(x)$ pravi podskup konveksnog skupa $\pi^{\alpha+1}(x)$. Kako dodavanje novih elemenata skupu $\pi^0(x) = \{x\}$ možemo raditi najvixe onoliko puta koliko ih ima u A, to za $\alpha = |A|$ mora važiti $\pi^\alpha(x) = \pi^{\alpha+1}(x)$. Tada će za svako $\beta > \alpha$ takodje važiti $\pi^\alpha(x) = \pi^\beta(x)$. Kontradikcija

□

Definicija 3.2. Neka je π kondenzacija i $\mathcal{A}$ linearno uredjenje. Za najmanji ordinal α takav da važi $\pi^\alpha(x) = \pi^{\alpha+1}(x)$ za svako $x \in A$, kažemo da je π-rang uredjenja $\mathcal{A}$ i obeležavamo sa $R_\pi(\mathcal{A})$.

Tako je na primer $R\pi_K(\omega^\alpha) = \alpha$.

4 Diskretna kondenzacija

Primeri kondenzacija koje smo naveli u prethodnom poglavlju su definisane beskonačnim konjukcijama (disjunkcijama) pa su invarijante izomorfizma, ali nisu definabilne formulom. U ovom poglavlju ćemo opisati jednu definabilnu kondenzaciju. Neka je $\mathcal{A} = (A, <)$ linearno uredjenje. Definišimo:

$$xEy \text{ ako i samo ako je } x \leqslant y \text{ i interval } [x, y] \text{ diskretno uredjen.}$$
$$x \sim y \text{ ako i samo ako } xEy \text{ ili } yEx.$$

U ostatku rada će $\sim$ biti rezervisano za upravo definisanu relaciju.

Lema 4.1. Relacija $x \sim y$ je definabilna relacija ekvivalencije čije su klase konveksne.

Dokaz: Formulom prvog reda se može reći da svaki element izmedju x i y ima neposrednog prethodnika i neposrednog sledbenika, da x ima neposrednog sledbenika i da y ima neposrednog prethodnika. Stoga je E, a time i $\sim$ definabilna relacija.

Relacija $\sim$ je očigledno refleksivna i simetrična. Da bi se pokazalo da je tranzitivna, dovljno je pokazati da ako je xEy i yEz, onda mora biti xEz. Svaka unutrašnja tačka intervala $[x, z]$ je ili unutrašnja tačka intervala $[x, y]$ ili je unutrašnja tačka intervala $[y, z]$ ili je baš y. U prva dva slučaja takva tačka ima i neposrednog prethodnika i neposrednog sledbenika jer je xEy i yEz, tj. $[x, y]$ i $[y, z]$ su diskretno uredjeni, u trećem slučaju y ima neporednog prethodnika jer je $[x, y]$ diskretan interval i neposrednog sledbenika jer je $[y, z]$ diskretan.

Neka je I jedna klasa ekvivalencije relacije $\sim$ i neka su $a, b \in I$. Bez umanjenja opštosti, neka je $a < b$. Tada je $[a, b]$ diskretno uredjen interval, pa je za svako $d \in [a, b]$ interval $[a, d]$ diskretno uredjen, pa je i d u istoj klasi kao i a, tj. $d \in I$. Dakle, $[a, b] \subseteq I$, pa je I konveksan skup.
qed

Lema 4.2. Ako je $\mathcal{A}$ je rasuto uredjenje, onda je $\mathcal{A}/_\sim$ rasuto.

Dokaz: Neka $\mathcal{A}/_\sim$ nije rasuto. To znači da u sebi kao podstrukturu sadrži kopiju uredjenja $(\mathbb{Q}, <)$. Neka je ta kopija racionalnih brojeva $(\{a_q/_\sim \mid q \in \mathbb{Q}\}, <)$. Tada je $(\{a_q \mid q \in \mathbb{Q}\}, <)$ podstruktura uredjenja $\mathcal{A}$ izomorfna uredjenju racionalnih brojeva.

□

Da obrnuto ne mora da važi pokazuje sledeći primer: Neka je
$\mathcal{A} = (\mathbb{Z} \times \mathbb{Q}) \times \omega$. Očigledno je da $\mathcal{A}$ nije rasuto, dok $\mathcal{A}/_\sim \cong \omega$ to jeste.

Kako su klase relacije $\sim$ diskretna linearna uredjenja, za njih postoji tačno pet mogućnosti: Pet tipova klasa kojima pridružujemo unarne predikate:

$P_0(x)$: ako je klasa $[x]$ oblika $\omega + \mathbb{Z} \times L + \omega^*$

$P_1(x)$: ako je klasa $[x]$ oblika $\omega + \mathbb{Z} \times L$

$P_2(x)$: ako je klasa $[x]$ oblika $\mathbb{Z} \times L + \omega^*$

$P_3(x)$: ako je klasa $[x]$ oblika $\mathbb{Z} \times L$

$P_{3+n}(x)$: ako je klasa $[x]$ oblika $\mathbf{n}$.

Predikati P_i su definabilni za $i > 0$, dok P_0 definiše beskonačna konjunkcija. U ostatku teksta, kad struktura ima medjusobno disjunktne unarne predikate R_i, često ćemo za njih reći da su boje i umesto da kažemo da važi $R_i(a)$, reći ćemo da je element a obojen bojom R_i; slično i za $P_i(a)$.

Dakle uredjenje $\mathcal{A} = (A, <)$ se definabilno širi bojama P_i do strukture $\mathcal{A}' = (A, <, \sim, P_i)_{i\in\omega}$. Uvek će nam $\mathcal{A}'$ značiti baš ovako prošireno $\mathcal{A}$.

Takodje, uredjenje $\mathcal{A} = (A, <)$ se kondenzuje u $\mathcal{A}/_\sim = (A/_\sim, <)$, a njega možemo da definabilno da proširimo, tako što ćemo element $x/_\sim$, obojiti bojom kojom je obojen sam x, a to je tačno boja kojom su obojeni svi elementi klase $[x]$. Na taj način se dobija $\mathcal{A}'/_\sim = (A/_\sim, <, P_i)_{i\in\omega}$.

Za elemente kondenzacije, a time i klase originala, postoje neka prirodna ograničenja, koje dajemo u sledećoj lemi.

Lema 4.3. Ako klasa ima maksimalni element ne može joj slediti klasa koja ima minimalni element. Slično, klasi koja nema maksimalni element, ne može slediti klasa koja nema minimalni element.

Dokaz: Pokazaćemo da iza klase obojene bojom P_0 ne može slediti klasa obojena istom bojom, a ostali delovi se slično dokazuju.

Pretpostavimo da $x/_\sim$ ima neposrednog sledbenika $y/_\sim$ i da su oba obojeni bojom P_0. Tada je $[x]$ oblika

$$\omega + \mathbb{Z} \times L_1 + \omega^*,$$

za neko linearno uredjenje L_1 (ili $L_1 = \emptyset$) i $[y]$ je oblika

$$\omega + \mathbb{Z} \times L_2 + \omega^*,$$

za neko linearno uredjenje L_2 (ili $L_2 = \emptyset$).

Onda je $[x] + [y]$ oblika $\omega + \mathbb{Z} \times L_1 + \omega^* + \omega + \mathbb{Z} \times L_2 + \omega^*$, t.j. oblika

$$\omega + \mathbb{Z} \times (L_1 + \mathbf{1} + L_2) + \omega^*,$$

što znači da su x i y u istoj, a ne u dve različite klase ekvivalencije. Kontradikcija. □

Napomena 4.1. Iz leme 4.3 sledi kakav je raspored dozvoljen (a kakav zabranjen), pa je isforsirano kakvog levog susednog ne sme imati element kondezacije obojen bojom P_i.

Teorema 4.1. Ako je $\mathcal{B} = (B, <, R_i)_{i\in I}$, pri čemu su R_i disjunktni unarni predikati i I je ω ili konačan ordinal, tada postoji linearno uredjenje $\mathcal{A}$, takvo da je $\mathcal{B}$ interpretabilno u njemu.

Dokaz: Bez umanjena opštosti možemo pretpostaviti da je svaki element iz $\mathcal{B}$ obojen nekom bojom, jer u suprotnom možemo boju R_i preoznačiti u boju Q_{i+1}, a one koji nisu obojeni označiti sa Q_0. Konstruišimo uredjenje $\mathcal{A}$: Za tačku $b \in B$ obojenu bojom R_n neka je $\mathcal{L}_b$ uredjenje $\mathbb{Z} + (\mathbf{n}+\mathbf{1}) + \mathbb{Z}$. Neka je $\mathcal{A} = \sum_{b\in\mathcal{B}} \mathcal{L}_b$. Pokazaćemo da je struktura $\mathcal{B}$ interpretabilna u uredjenju $\mathcal{A}$.

Pogledajmo prvo šta su klase ekvivalencije $\sim$ u uredjenju $\mathcal{A}$. Očigledno za svako $b \in B$, postoji jedna konačna klasa u uredjenju $\mathcal{A}$. Levo od svake konačne klase je tačno jedna od sledeće dve: $\mathbb{Z} \times 2$ ili $\mathbb{Z}$. Tačnije, ako u uredjenju $\mathcal{B}$ element b nije imao neposrednog prethodnika i bio je obojen bojom R_i, onda njemu odgovara klasa koja ima $i+1$ element i čiji neposredni prethodnik je klasa oblika $\mathbb{Z}$. Ako je element b imao neposrrednog prethodnika, onda je neposredni prethodnik klase $i+1$ klasa oblika $\mathbb{Z} \times 2$. Slično, sa desne strane svake konačne klase ekvivalencije $\sim$ je ili klasa oblika $\mathbb{Z} \times 2$ ili $\mathbb{Z}$. U svakom slučaju, levo i desno od svake konačne klase je klasa u boji P_3. Uredjenje $\mathcal{A}$ nema konvekse podskupove boja P_0, P_1 i P_2, pa se kondenzuje u

$$\mathcal{A}'/_{\sim} = (A/_{\sim}, <, P_j)_{j\in\{3, i+4 \,|\, i\in I\}}.$$

Očigledno da ako svako $b \in \mathcal{B}$ koje je obojeno bojom R_i prefarbamo u boju P_{i+4}, izmedju svaka dva susedna elementa dodamo jedan obojen bojom P_3, ispred svakog levog graničnog i iza svakog desnog graničnog dodamo jedan element obojen bojom P_3, dobijamo uredjenje izomorfno sa $\mathcal{A}'/_{\sim}$. Zato je relativizacija strukture $\mathcal{A}'/_{\sim}$ na formulu $\neg P_3(x)$ izomorfna sa strukturom $\mathcal{B}'$.

Sa $R(\mathcal{A})$ označavamo kondenzacioni rang uredjenja $\mathcal{A}$ pri diskretnoj kondezaciji. Sledeću teoremu navodimo bez dokaza.

Teorema 4.2. Ako je $\mathcal{A} = (A, <)$ rasuto linearno uredjenje sa malom teorijom tada je $R(\mathcal{A}) < \omega$.

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Piunikhin–Salamon–Schwarz isomorphisms and symplectic invariants obtained using cobordisms of moduli spaces

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Abstract

We give a brief survey of results considering the Piunikhin–Salamon–Schwarz isomorphism in Lagrangian Floer homology groups generated by Hamiltonian orbits starting at the zero section and ending at the conormal bundle. We discuss the PSS isomorphism in Lagrangian Floer homology groups for the open subset. Also we discuss some properties of the absolute and relative spectral invariants.

1 Introduction

Spectral invariants have been used to prove some important properties in symplectic topology (see [5, 18, 25, 26]). Non–degeneracy of Hofer's distance on the group of Hamiltonian diffeomorphisms $\mathcal{G}$, existence of partial quasi–morphisms on $\mathcal{G}$ and partial quasi–states are just few of them. One way to construct spectral invariants in Floer theory is using a certain isomorphism between Morse homology and Floer homology, Piunikhin–Salamon–Schwarz isomorphism (which we abbreviate to PSS). This article is organized as follows. In Sections 2 and 3 we recall the definitions of Morse and Floer homologies. In Section 4 we review the known results on PSS homomorphisms. Section 5 contains the definitions of symplectic invariants. New results are presented in Sections 6, 7 and 8.

2 Morse homology

Let us take a compact smooth manifold M. We say that a smooth function

$$f : M \to \mathbb{R}$$

is a Morse function if all critical points of f are nondegenerate. Geometrically, it means that a differential of a function f is transversal to the zero section in the cotangent bundle

$$df \pitchfork O_M \subset T^*M.$$

Using the critical points of a Morse function we can describe the topology of a manifold (see [17] for more details). For a generic choice of a Morse function and Riemannian structure on M we can define the Morse homology $HM_*(f:M)$. For generators of a chain complex we take critical points of a function f,

$$CM_* = \mathbb{Z}_2\langle p \in \mathrm{Crit}(f)\rangle.$$

It is graded by the Morse index of a critical point. Differential

$$\partial_M : CM_k(f:M) \to CM_{k-1}(f:M)$$

counts the number of gradient trajectories (see [24] for details). Morse homology groups, $HM_*(f:M)$ are isomorphic to the singular homology groups $H_*(M;\mathbb{Z}_2)$, see [16, 24] (we will identify them sometimes).

3 Floer homology and action functional

Floer homology is an infinite–dimensional analogue of Morse homology (see [6]–[13]). In Morse homology case we observe critical points of a function defined on a finite–dimensional manifold. In Floer homology case we are interested in critical points of an action functional, which is now a function defined on an appropriate infinite–dimensional space of paths.

Let us take a closed symplectic manifold (P,ω). For a smooth Hamiltonian $H:[0,1]\times P \to \mathrm{R}$ we define the Hamiltonian vector field X_H with

$$\iota_{X_H}\omega = dH.$$

Family of Hamiltonian diffeomorpshisms ϕ_H^t is a solution of a differential equation with initial condition

$$\frac{d}{dt}\phi_H^t(x) = X_H(\phi_H^t(x)),$$
$$\phi_H^0 = \mathrm{Id}\,.$$

We say that a submanifold L of a symplectic manifold P is a Lagrangian submanifold if $\dim L = \frac{1}{2}\dim M$ and $\omega|_{TL} = 0$. We consider a closed smooth Lagrangian submanifold L such that

$$\omega|_{\pi_2(P,L)} = 0, \qquad \mu|_{\pi_2(P,L)} = 0, \tag{1}$$

where μ is the Maslov index (see [21, 22]).

3.1 Hamiltonian Floer homology

Let us define the space of contractible loops in P

$$\mathcal{P}(P) = \{\gamma \in C^\infty(S^1,P) \mid [\gamma] = 0 \in \pi_1(P)\}.$$

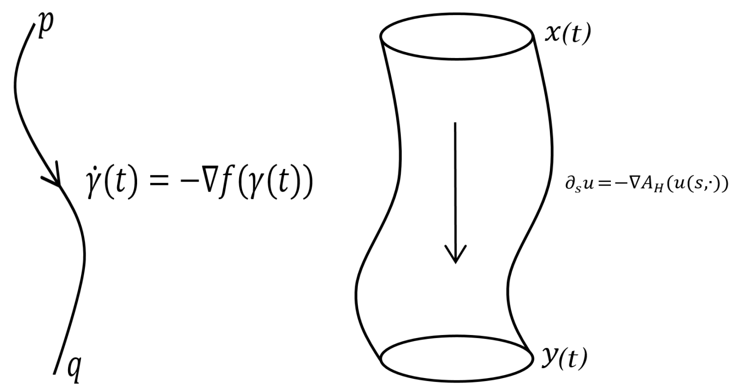


Figure 1: *Decreasing of function along gradient trajectory and action functional along holomorphic cylinder*

The action functional for contractible loops is defined as

$$\mathcal{A}_H(\gamma) = -\iint_{\mathbb{D}^2} \widetilde{\gamma}^*\omega - \int_{\mathbb{S}^1} H(\gamma(t), t)\, dt.$$

Here $\widetilde{\gamma} : \mathbb{D}^2 \to P$ is any extension of γ to the unit disc. From the second topological assumption in (1) it follows that

$$\omega|_{\pi_2(P)} = 0,$$

thus the action functional is well defined (it does not depend on an extension $\widetilde{\gamma}$). Its (finite number of) critical points

$$\operatorname{Crit}\mathcal{A}_H$$

are Hamiltonian periodic orbits. Hamiltonian Floer chain complex is $\mathbb{Z}_2$ vector space over $\operatorname{Crit}\mathcal{A}_H$. It is graded by Conley–Zehnder index (see [23] for definition) and a boundary operator counts the number of the perturbed holomorphic cylinder that connects two elements of $\operatorname{Crit}\mathcal{A}_H$ (see Figure 1). Homology of this complex is Hamiltonian Floer homology of generic smooth Hamiltonian, $HF_*(H)$. Since the action functional decreases along holomorphic cylinder filtered Hamiltonian Floer homology is well defined, $HF_*^\lambda(H)$.

3.2 Lagrangian Floer homology

Generators of Lagrangian Floer homology are intersection points of L and $\phi_H^1(L)$ (assuming that these Lagrangian submanifolds are intersecting transversally, $L \pitchfork \phi_H^1(L)$). We can see these generators as critical points of an action functional, a_H. Domain of a_H is a space of paths

$$\mathcal{P}(L) = \{x \in C^\infty([0,1], P) \mid x(0), x(1) \in L,\, [x] = 0 \in \pi_1(P, L)\},$$

and at $x \in \mathcal{P}(L)$ it takes value

$$a_H(x) = -\iint_{D^2_+} h^*\omega - \int_0^1 H(x(t), t)\, dt.$$

Here, h is a map from the upper half–disc D^2_+ to P that restricts to x on the upper half–circle. Because of the first assumption in (1) $a_H(x)$ does not depend on an extension h. A boundary operator ∂ of Lagrangian Floer complex counts the number of perturbed holomorphic discs with a boundary on a mentioned submanifold. Chain complex is graded by relative Maslov index. Homology of this chain complex is called Floer homology for Lagrangian intersections and it is denoted by $HF_*(L, \phi^1_H(L))$.

We can define filtered complex as

$$CF^\lambda_* = \mathbb{Z}_2\langle x \in \operatorname{Crit} a_H \mid a_H(x) < \lambda\rangle.$$

Since a_H decreases along holomorphic strip ∂ descends to a boundary operator on a filtered complex. Homology of a filtered complex is called filtered Lagrangian Floer homology and it is denoted by $HF^\lambda_*(L, \phi^1_H(L))$.

Floer homology for Lagrangian intersections is a generalization of Floer homology for periodic orbits. We know that diffeomorphism $\phi : P \to P$ preserves the symplectic form if and only if its graph Γ_ϕ is a Lagrangian submanifold in $(P \times P, \omega \oplus -\omega)$. Periodic orbits of ϕ are in one–to–one correspondence with intersection points of Lagrangian submanifolds Γ_ϕ and the diagonal $\Delta = \{(x, x) \mid x \in P\}$.

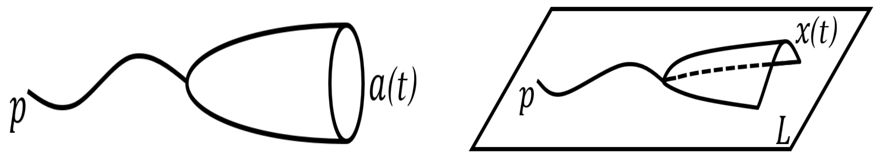
4 PSS isomorphism

Floer [13] proved that when $P = T^*M$ is the cotangent bundle over a compact manifold M and $L = O_M$ is the zero section, Floer homology is isomorphic to the singular homology of M. An extension of any C^2-small Morse function $f : M \to \mathbb{R}$ to the cotangent bundle gives a Hamiltonian $H_f : T^*M \to \mathbb{R}$ such that the intersection points $O_M \cap \phi^1_{H_f}(O_M)$ are in one–to–one correspondence with critical points of f. So appropriate gradient trajectories of f are in one–to–one correspondence with holomorphic strips of H_f. This construction was generalized by Poźniak [20]. He considered Floer homology $HF_*(O_M, \nu^*N)$ and proved that it is isomorphic to the singular homology of N. Here $\nu^*N \subset T^*M$ is the conormal bundle of a closed submanifold $N \subset M$. This idea was further generalized by Kasturirangan and Oh [14] to an open subset $U \subset M$. Let us denote by $S^{\alpha\beta}$ isomorphism between Floer homologies for different Hamiltonians,

$$S^{\alpha\beta} : HF_*(H_\alpha) \to HF_*(H_\beta),$$

(it is well known that these groups do not depend on the Hamiltonian). Let $T^{\alpha\beta}$ denotes isomorphism between Morse homologies for two different Morse functions,

$$T^{\alpha\beta} : HM_*(f_\alpha : M) \to HM_*(f_\beta : M),$$



a) b)

Figure 2: *Combined type object that defines PSS*

(see [24] for details). Since $S^{\alpha\beta}$ and $T^{\alpha\beta}$ are defined by counting the number of solutions of different type of equations it is not obvious whether the diagram

$$\begin{array}{ccc} HF_*(H^\alpha) & \xrightarrow{S^{\alpha\beta}} & HF_*(H^\beta) \\ \uparrow & & \uparrow \\ HM_*(f^\alpha) & \xrightarrow{T^{\alpha\beta}} & HM_*(f^\beta) \end{array} \tag{2}$$

commutes (vertical arrows are previously described isomorphisms). This question is positively answered if we use different type of isomorphisms as vertical arrows (see theorems below).

Theorem 4.1. *[19] There exists an isomorphism*

$$PSS : HM_*(f : P) \to HF_*(H),$$

between the Morse homology of a Morse function $f : P \to \mathbb{R}$ and the Hamiltonian Floer homology such that the diagram (2) commutes.

In order to construct isomorphism in Theorem 4.1 Piunikhin, Salamon and Schwarz observed combined type objects (see Figure 2a). Similar construction for holomorphic discs with boundary condition on the zero section (see Figure 2b) in the cotangent bundle was carried out by Katić and Milinković.

Theorem 4.2. *[15] There exists an isomorphism*

$$PSS : HM_*(f : M) \to HF_*(O_M, \phi^1_H(O_M))$$

between the Morse homology of a Morse function $f : M \to \mathbb{R}$ and the Lagrangian Floer homology of the zero section in the cotangent bundle T^*M. For this isomorphism diagram (2) commutes.

Albers constructed PSS–type homomorphism in more general case (which is not necessary an isomorphism, see [1]).

Theorem 4.3. *[1] There exists Lagrangian PSS homomorphism*

$$PSS : HM_*(f : L) \to HF_*(L, \phi^1_H(L)).$$

5 Symplectic invariants

Using the existence of PSS isomorphism we can define symplectic invariants and prove some of their properties.

Definition 5.1. For $\alpha \in H_*(P)\backslash\{0\}$ we define a *symplectic invariants for periodic orbits* as

$$\rho(\alpha, H) := \{\lambda \in \mathbb{R} \,|\, PSS(\alpha) \in \mathrm{Im}(\iota_*^\lambda)\}.$$

Here, ι_*^λ is a homomorphism induced by an inclusion

$$\iota^\lambda : CF_*^\lambda(H) \to CF_*(H).$$

Definition 5.2. *Symplectic invariants in Lagrangian case* are defined as

$$c(\beta, H) := \inf\{\lambda \in \mathbb{R} \,|\, PSS(\beta) \in \mathrm{Im}(j_*^\lambda)\},$$

where $\beta \in H_*(L) \setminus \{0\}$. A homomorphism j_*^λ is induced by an inclusion

$$j^\lambda : CF_*^\lambda(L, \phi_H^1(L)) \to CF_*(L, \phi_H^1(L)).$$

In Definition 5.1 we used PSS isomorphism from Theorem 4.1 and in Definition 5.2 one defined in Theorem 4.3.

6 Comparison of symplectic invariants

We can compare symplectic invariants in periodic and Lagrangian case using the homomorphisms defined via "chimneys" (see [3] for details).

Theorem 6.1. *For $\beta \in H_*(L) \setminus \{0\}$ it holds*

$$\rho(\iota_*(\beta), H) \leq c(\beta, H).$$

Here, $\iota_* : H_*(L) \to H_*(P)$ is a homomorphism obtained from an inclusion $i : L \to P$.

Theorem 6.2. *For $\alpha \in H_*(P) \setminus \{0\}$ it holds*

$$\rho(\alpha, H) \geq c(i_!(\alpha), H).$$

Here, $i_! = PD^{-1} \circ \iota_* \circ PD$ is a homomorphism defined by Poincaré duality map and an inclusion.

We construct a product

$$\cdot : HF_*(H_1) \otimes HF_*(L, \phi_{H_2}^1(L)) \to HF_*(L, \phi_{H_3}^1(L))$$

and prove a subadditivity of invariants with respect to this product (see [3]).

Theorem 6.3. *For $a \in HF_*(H_1)$, $b \in HF_*(L, \phi_{H_2}^1(L))$ such that $a \cdot b \neq 0$ it holds*

$$c(PSS^{-1}(a \cdot b), H_1 \sharp H_2) \leq \rho(PSS^{-1}(a), H_1) + c(PSS^{-1}(b), H_2).$$

7 Conormal bundle

Let us take a closed submanifold $N \subset M$. Its conormal bundle, ν^*N is a Lagrangian submanifold in T^*M. Although the intersection $O_M \cap \nu^*N$ is not transversal we can define Floer homology that counts Hamiltonian orbits that start at the zero section and end at the conormal bundle (see [8,18]). We denote this homology by $HF_*(O_M, \nu^*N)$. It turns out that these Floer homologies are isomorphic to Morse homologies of a Morse function defined on N.

Theorem 7.1. *There exists an isomorphism*

$$PSS : HF_*(O_M, \nu^*N) \cong HM_*(f : N)$$

such that the diagram (2) commutes.

Details on the proof could be found in [2]. We also defined a product on Floer homology and proved that conormal symplectic invariants

$$l(\alpha, H) := \inf\{\lambda \,|\, PSS(\alpha) \in \mathrm{Im}(\iota_*^\lambda)\},\ \alpha \in H_*(N) \setminus \{0\},$$

are subadditive with respect to this product. We found an upper bound for all conormal symplectic invariants $l(\cdot, H)$.

8 Open subset

Let us take an open subset, $U \subset M$. Lagrangian Floer homology for an open subset, $HF(H, U : M)$, in the cotangent bundles, T^*M, was introduced by Kasturirangan and Oh in as a part of a project of "quantization of Eilenberg–Steenrod axioms" (see [14]). This homology is defined as a direct limit of Floer homologies of approximations.

Theorem 8.1. *There exists PSS–type isomorphism*

$$PSS_U : HM_*(f : U) \to HF_*(H, U : M)$$

such that diagram (2) commutes.

This theorem is proven in [4]. Invariants are defined similarly as in the previous cases,

$$c_U(\alpha, H) = \inf\{\lambda \,|\, PSS_U(\alpha) \in \mathrm{Im}(\iota_*^\lambda)\},$$

for $\alpha \in H_*(U)$, $\alpha \neq 0$. We define a module structure product on $HF(H, U : M)$ and prove the triangle inequality for invariants with respect to this product. We also prove the continuity of these invariants and compare them with spectral invariants for periodic orbit case in T^*M (see [4] for details).

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ČETVRTA MATEMATIČKA KONFERENCIJA REPUBLIKE SRPSKE
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Rješavanje Sturm-Liouvilleovog problema sa konstantnim kašnjenem metodom upucavanja

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Apstrakt

Metoda upucavanja se već dugi niz koristi za numeričko rješavanje diferencijalnih jednačina, kao i za rješavanje Sturm-Liouvilleovog problema. Međutim, kod klasičnog Sturm-Liouvilleovog problema, sve svojstvene vrijednosti i svojstvene funkcije su realne, dok se kod Sturm-Liouvilleovog problema sa kašnjenjem, pored realnih svojstvenih vrijednosti pojavljuju i kompleksne svojstvene vrijednost, a samim time i kompleksne svojstvene funkcije. U ovom radu proširujemo primjenu ove metode na određivanje svojstvenih vrijednosti i svojstvenih funkcija (kako realnih, tako i kompleksnih) Sturm-Liouvilleove jednačne sa konstantim kašnjenjem pri graničnim uslovima Dirichletovog tipa.

1 Uvod

Posmatraćemo Sturm-Liovilleovu jednačinu sa konstantnim kašnjenjem

$$-y''(x) + q(x)y(x - \tau) = \lambda y(x), \tag{1}$$

pri čemu ćemo za $q(x) \in L_2[0, \pi]$ uzimati realnu funkciju koja ne oscilira brzo, a za kašnjenje $\tau \in \mathbb{R}$ i $\tau > 0$. Pored ove jednačine posmatraćemo i granične uslove Dirichletovog tipa

$$y(0) = 0, \quad y(\pi) = 0. \tag{2}$$

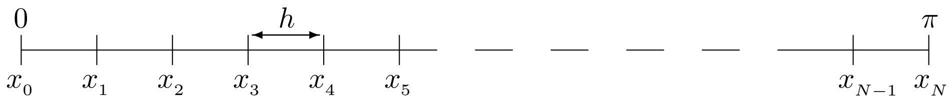
Netrivijalna rješenja jednačine (1) uz posmatrane granične uslove nazivaju se svojstvene funkcije, a λ za koje jednačina (1) ima netrivijalna rješenja nazivaju se svojstvene vrijednosti. Kod Sturm-Liovilleovog problema bez kašnjenja, pokazano je da su sve svojstvene vrijednosti i sve svojstvene funkcije realne.

Međutim, kod Sturm-Liovilleove jednačine sa kašnjenjem svojsvene vrijednosti ne moraju biti realne, pa samim time i pripadajuće svojstvene funkcije ne moraju biti realne.

2 Određivanje realnih svojstvenih vrijednosti

Pokazaćemo kako koristeći metodu upucavanja možemo približno odrediti realne svojstvene vrijednosti i pripadajuće svojstvene funkcije.

Metoda upucavanja sastoji se u sljedećem. Izvršićemo podjelu segmenta $[0, \pi]$ na N dijelova (koji ne moraju biti jednaki, ali ćemo jednostavnosti radi smatrati da su jednaki i da je $h = \frac{\pi}{N}$), tačkama x_i, $i = \overline{1, N}$, pri čemu je $x_0 = 0$ i $x_N = \pi$.



Također, ćemo smatrati da je τ višetrukost od h. Neka je npr. $\tau = w \cdot h$, $w \in \mathbb{N}$. Sada će biti $x_i - \tau = ih - wh = (i - w)h = x_{i-w}$.

Uvodeći smjenu $y' = p$ u jednačinu (1) mi ovu jednačinu drugog reda u kojoj se ne pojavljuje prvi uzvod, možemo napisati u obliku sistema

$$\begin{aligned} p'(x) &= q(x)y(x-\tau) - \lambda y(x) \\ y'(x) &= p(x) \end{aligned}$$

Mi ćemo počev od tačke $x_0 = 0$ riješiti gornji sistem. $y(x_0) = y(0) = 0$ nam je poznato, međutim da bi riješili sistem potrebno nam je još i $p(x_0) = p(0)$. Sljedeća lema nam daje ovu vrijednost.

Lema 2.1. Ako je $\tilde{y}(x)$ rješenje jednačine (1), tada je i $C\tilde{y}(x)$ također njeno rješenje.

Dokaz, ove leme je tivijalan, jer je jednačina 2.1 homogena diferencijalna jednačina, međutim ona nam daje za pravo da za $p(0)$ odaberemo proizvoljnu vrijednost. Uzimajući $p(0) = 0$ dobijamo trivijalno rješenje $y(x) \equiv 0$. Međutim, nas interesuju netrivijalna rješenja, pa ćemo proizvoljno $p(0)$ odabrati koristeći sljedeću lemu.

Lema 2.2. Jednačina (1) sa graničnim uslovom $y(0) = 0$ ekvivalentna je sa

$$y(x) = \frac{1}{z} \int_0^x q(t)y(x-\tau) \sin z(x-t)dt + C_1 \sin zx.$$

Odavde je

$$y'(x) = \frac{1}{z}\left(q(x)y(x-\tau)\sin zx \cos zx + z\cos zx \int_0^x q(t)y(x-\tau)\cos zt dt - \right.$$
$$\left. -q(x)y(x-\tau)\cos zx \sin zx + z \sin zx \int_0^x q(t)y(x-\tau)\sin zt dt \right) + C_1 z \cos zx.$$

Uzimajući da je $C_1 = 1$ i $x = 0$ dobijamo da je $y'(0) = z$.

Pretpostavimo sada da tražimu n-tu svojstvenu vrijednost i svojstvenu funkciju. Za λ ćemo uzeti da je približno jednako (vidjeti [5])

$$\lambda = n^2 + \frac{\cos(\tau n)}{\pi} \int_{\tau}^{\pi} q(t)dt - \frac{1}{\pi} \int_{\tau}^{\pi} q(t)\cos(2t - \tau)dt.$$

Dakle, sada kada imamo početne uslove možemo koristeći se nekom od metoda riješiti jednačnu (1) uz početne uslove. Npr. koristeći Eulerov metod za numeričko rješavanje diferencijalnih jednačina imamo:

$$p(x_{i+1}) = p(x_i) + h \cdot \left[q(x_i)y(x_{i-w}) - \lambda y(x_i)\right]$$
$$y(x_{i+1}) = y(x_i) + hp(x_i)$$

Stavimo da je $E(\lambda) = y(x_N) - y(\pi) = y(x_N) - 0 = y(x_N)$ Sada možemo primjeniti sljedeći algoritam:

- Za λ_1 uzimamo približnu vrijednost svojstvene vrijednosti.
- Za λ_2 uzimamo približnu vrijednost svojstvene vrijednosti različitu od λ_1.
- λ_3 određujemo npr. metodom sekante na sljedeći način:

$$\lambda_3 = \lambda_2 - \frac{E(\lambda_2) - E(\lambda_1)}{\lambda_2 - \lambda_1} E(\lambda_2)$$

- Postupak ponavljamo za λ_k dok promjena ne bude dovoljno mala

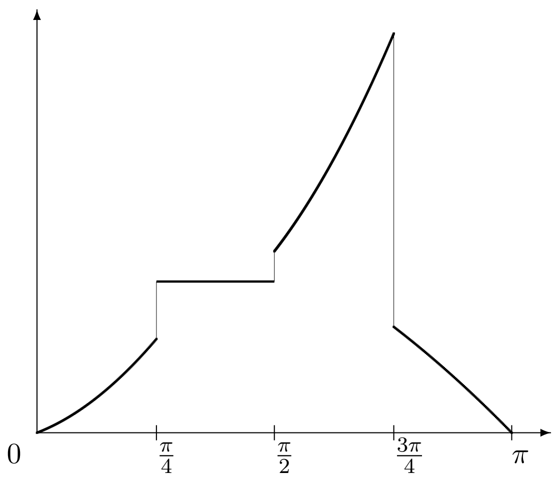
Dobre strane algoritma su: Preciznost zavisi od metode za rješavanje diferencijalne jednačine, Preciznost sporo opada za svojstvenom vrijednošću, Uz svojsvenu vrijenost dobije se i svojsvena funkcija, dok su ne tako dobre strane algoritma, svaka svojstvena vrijednost se mora tražiti posebno, svojstvene funkcije su oscilirajuće sa približno k ekstrema pa numerički algoritam za rješavanje diferencijalne jednačine gubi na preciznosti.

Demonstrirajmo ovo na jednom primjeru. Za potencijal uzmimo funkciju

$$SF1(x) = \begin{cases} x^2 & , x \in [0, \frac{\pi}{4}] \\ 1 & , x \in [\frac{\pi}{4}, \frac{\pi}{2}] \\ \frac{e^x}{4} & , x \in [\frac{\pi}{2}, \frac{3\pi}{2}] \\ \sin(x) & , x \in [\frac{3\pi}{2}, \pi] \end{cases}$$

čiji je grafički prikaz dat slikom 1

Za broj podjela uzimamo 314 159 što nam daje približnu vrijednost za $h \approx 0,000\,01$ dok za kašnjenje uzimamo $\tau \approx 1$, tj. $\tau = \frac{\pi}{314\,159} 10000$. U tabeli 1 u prvoj koloni se nalazi redni broj svojstvene vrijednosti, u drugoj svojstvene vrijednosti



Slika 1: Skica funkcije $SF1(x)$

dobijene metodom upucavanja, u trećoj svojstvene vrijednosti dobijene matričnom metodom, a četvrtoj asimptotske svojstvene vrijednosti.

Napomenimo da iz [5] slijedi da se svojstvene vrijednosti ponašaju u skladu sa asimptotikom

$$\lambda_n \approx n^2 + \frac{\cos(\tau n)}{\pi} \int_{\tau}^{\pi} q(t)dt - \frac{1}{\pi} \int_{\tau}^{\pi} q(t)\cos(2t-\tau)dt.$$

Primjećujemo da u ovom slučaju su sve svojstvene vrijednosti realne i da su veće svojstvene vrijednosti približno jednake asimptotici.

3 Određivanje kompleksnih svojstvenih vrijednosti

Ako za potencijal uzmemo funkciju $q(x) = e^x$, a za kašnjenje $\tau = 1.7$ primjenom prethodne metode dobijamo sljedeći niz svojstvenih vrijednosti $9,3084194073$, $32,6709624001, 51,8602143099, 98,5577886424, \dots$, koje upoređujući ih sa asimptotikom predstavljaju treću, šestu i sedmu i desetu svojstvenu vrijednost. Dakle, nedostaju nam prva-druga, četvrta-peta, osma-deveta svojstvena vrijednost.

Pretpostavimo da su svojsvene vrijednosti koje nedostaju kompleksne i zapišimo iz u obliku $\lambda = \lambda_R + i\lambda_I$ i $y(x) = y_R(x) + iy_I(x)$ pa uvrštavajući u

$$-y''(x) + q(x)y(x-\tau) = \lambda y(x),$$

dobijamo

$$-(y''_R(x) + iy''_I(x)) + q(x)(y_R(x-\tau) + iy_I(x-\tau)) = (\lambda_R + i\lambda_I)(y_R(x) + iy_I(x)),$$

#	λ	λ_N	$\tilde{\lambda}$	ϵ
1	2,311763630	2,311706667	1,422488336	0,000000000
2	3,125569842	3,123531021	3,794128003	0,000000000
3	8,358472794	8,357864037	8,689041123	0,000000001
4	15,499976452	15,502890412	15,722101640	0,000000005
5	25,151810498	25,154595692	25,089014320	0,000000014
6	36,698741195	36,701902005	36,361244615	0,000000029
7	49,564872249	49,560451632	49,266460812	0,000000055
8	63,939698106	63,932414226	63,973634869	0,000000093
9	80,290742730	80,288625317	80,656317537	0,000000150
10	99,411430398	99,415015770	99,696021824	0,000000229
100	10000,644881400	10000,259661442	10000,314382795	0,002291672
200	40000,427466831	39974,226567527	40000,177292881	0,036666862
298	88803,358386810	88510,993024225	88803,670633695	0,180726045
299	89400,402954226	89102,275322440	89400,688120480	0,183164146
300	90000,161112526	89695,869455141	89999,991192361	0,185626833
301	90601,947280797	90291,146127565	90601,303660572	0,188114271
302	91205,036202636	90887,664522780	91204,335844614	0,190626624

Tabela 1: Rezultati dobijenu u prvom primjeru

Nakon sređivanja dobijamo

$$-y_R''(x) - iy_I''(x)) + q(x)y_R(x-\tau) + iq(x)y_I(x-\tau) = \\ = \lambda_R y_R(x) + i\lambda_I y_R(x) + i\lambda_R y_I(x) - \lambda_I y_I(x),$$

Izjednačavajući realne i imaginarne dijelove dobijamo sistem

$$\begin{aligned} -y_R''(x) + q(x)y_R(x-\tau) &= \lambda_R y_R(x) - \lambda_I y_I(x) \\ -y_I''(x) + q(x)y_I(x-\tau) &= \lambda_R y_I(x) + \lambda_I y_R(x), \end{aligned}$$

tj. stavljajući $p_R' = y_R'$ i $p_I' = y_I'$ dobijamo sistem

$$\begin{aligned} p_R'(x) &= q(x)y_R(x-\tau) - \lambda_R y_R(x) + \lambda_I y_I(x) \\ p_R'(x) &= y_R(x) \\ p_I'(x) &= q(x)y_I(x-\tau) - \lambda_R y_I(x) - \lambda_I y_R(x) \\ p_I'(x) &= y_I(x) \end{aligned}$$

koji se numerički riješava preko sistema

$$\begin{aligned} p_R(x_{i+1}) &= p_R(x_i) + h\left[q(x_i)y_R(x_{i-w}) - \lambda_R y_R(x_i) + \lambda_I y_I(x_i)\right] \\ y_R(x_{i+1}) &= y_R(x_i) + hp_R(x_i) \\ p_I(x_{i+1}) &= p_I(x_i) + h\left[q(x_i)y(x_{i-w}) - \lambda_R y_I(x_i) - \lambda_I y_R(x_i)\right] \end{aligned}$$

$$y_I(x_{i+1}) = y_I(x_i) + hp_I(x_i)$$

pri čemu za početne uslove uzimamo $y_R(0) = 0$, $y_I(0) = 0$, $p_R(0) = k$, $p_R(0) = k$,
Sada možemo krenuti u iterativni proces određivanje svojstvene vrijednosti.

- Za $\lambda_1 = \begin{pmatrix} \lambda_{R1} \\ \lambda_{I1} \end{pmatrix}$ uzimamo približnu vrijednost svojstvene vrijednosti.
- Za $\lambda_2 = \begin{pmatrix} \lambda_{R2} \\ \lambda_{I2} \end{pmatrix}$ uzimamo približnu vrijednost svojstvene vrijednosti.
- λ_3 određujemo npr. na sljedeći način (moguće je koristi i neki drugi metod za određivanje nula vektorske funkcije):

$$J = \begin{pmatrix} \frac{\partial E_R(\lambda_R)}{\partial \lambda_R} & \frac{\partial E_R(\lambda_R)}{\partial \lambda_I} \\ \frac{\partial E_I(\lambda_I)}{\partial \lambda_R} & \frac{\partial E_I(\lambda_I)}{\partial \lambda_I} \end{pmatrix}$$

$$\lambda_3 = \lambda_2 - J^{-1} \begin{pmatrix} E_R(\lambda_R) \\ E_I(\lambda_I) \end{pmatrix}$$

- Postupak ponavljamo za λ_k dok promjena ne bude dovoljno mala.

Sada za $q(x) = e^x$, uzmimo $N = 314\,159$ za broj podjela, što nam daje približnu vrijednost za $h \approx 0,000\,01$ dok za kašnjenje uzimamo $\tau \approx= 1.7$, tj. $\tau = \frac{\pi}{314159} \cdot 157\,079$. U tabeli 2 u prvoj koloni se nalazi redni broj svojstvene vrijednosti, u drugoj svojstvene vrijednosti dobijene metodom upucavanja, u trećoj svojstvene vrijednosti dobijene matričnom metodom, a četvrtoj asimptotske svojstvene vrijednosti.

Dakle, $\lambda_1 \approx 1,577005781 + i\,2,673316607$, $\lambda_4 \approx 21,531809100 + i\,3,9432489433$ i $\lambda_8 \approx 71,806437391 + i\,0,999188030$, pa su zaista prva, četvrta i osma svojstvena vrijednost kompleksne svojstvene vrijednosti. Napomenimo da je greška kod oređivanja kompleksnih svojstvenih vrijednosti veća nego kod određivanja realnih svojstvenih vrijednosti, jer se numerički rješava sistem od četiri diferencijalne jednačine.

4 Zaključak

U ovom radu pokazan je način za određivanje svojstvenih vrijednosti Sturm-Liouvilleovog problema sa konstantim kašnjenjem metodom upucavanja, bez obzira na to da li su svojstvene vrijednosti realne ili kompleksne. Pri tome je na dva konkretna primjera pokazana ova metoda, pri čemu je u drugom primjeru demonstrirano postojanje kompleksnih svojstvenih vrijednosti za Sturm-Liouvilleov problem sa konstantim kašnjenjem.

#	λ	λ_N	$\tilde{\lambda}$	ϵ
1	1,577005781	1,57887456	x	x
1i	**2,673316607**	**2,66650830**	x	x
3	9,308419407	9,31556254	13,723662625	0,000000001
4	21,531809100	21,51429291	x	x
4i	**3,9432489433**	**3,937466800**	x	x
6	32,670962400	32,70359912	25,477433265	0,000000029
7	51,860214309	51,88044110	60,094061531	0,000000055
8	71,806437391	71,78094879	x	x
8i	**0,999188030**	**0,73424580**	x	x
10	98,557788642	98,60305951	95,826131793	0,000000229
100	10005,604627914	10004,73114541	10013,498158024	0,002291672
200	40005,197362867	39977,21732008	40010,933013158	0,036666862
298	88798,536305550	88509,27193604	88794,011430397	0,180726045
299	89407,685800385	89108,01179285	89412,548346491	0,183164146
300	90004,288859502	89696,75236583	90007,012647369	0,185626833
301	90593,469717288	90284,83947264	90587,644546946	0,188114271
302	91202,134466681	90887,47735875	91200,428952075	0,190626624

Tabela 2: Rezultati dobijenu u drugom primjeru

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Weakly compatible mappings in Menger spaces and fixed point results

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Abstract

In this paper is proved some common fixed point theorems for weakly compatible mappings in Menger spaces using the common property (E.A). Some ilustrative examples are given to demonstrate the validity of main result.

1 Introduction and Preliminaries

In 1922. S. Banach proved the principal contraction result [1]. Since then, there have been published many works about fixed point theory for different kinds of contractions on some spaces such as: quasi-metric [2], cone metric spaces [3]-[5], convex metric spaces [6], partially ordered metric spaces [7]-[9], G-metric spaces [10]-[12], partial metric spaces [13, 14], quasi-partial metric spaces [15], fuzzy metric spaces[16, 17], Menger spaces [18]-[28],... All of these spaces, with or without ordering are generalization of metric spaces.
K. Menger [18] introduced the notion of probabilistic metric space in 1942 (shortly PM spaces) in which the concept of distance is considered to be statistical or probabilistic rather than deterministic. The notion of PM-space corresponds to situacion when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance.
Many mathematicians was proved several common fixed point theorems for contraction mappings in Menger spaces by using different notions: compatible mappings, weakly compatible mappings, property (E.A), common property (E.A),... [19]-[28]. In this paper is establish a common fixed point theorem for two pairs of weakly compatible mappings in Menger spaces using the common property (E.A). First we recall some definitions and known results in Menger spaces

Definition 1.1. [25] A triangular norm $*$ (shortly t- norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied:
(1) $a * 1 = 1$,
(2) $a * b = b * a$,
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,
(4) $a * (b * c) = (a * b) * c$.

Definition 1.2. [25] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. With $\Im$ we denote the set of all distribution function on $\mathbb{R}$, while H denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is nonempty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition 1.3. [25] The ordered pair $(X, \mathcal{F})$ is called a probabilistic metric space (PM-space) if X is nonempty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions:

(1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(2) $F_{x,y}(0) = 0$ for all $x, y \in X$,
(3) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and for all $t > 0$,
(4) $F_{x,z}(t) = 1, F_{x,y}(s) = 1 \Longrightarrow F_{x,y}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can always be realized as a probabilistic metric space defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $t > 0$.

Definition 1.4. [18] A Menger space $(X, \mathcal{F}, *)$ is a tripled where $(X, \mathcal{F})$ is PM-space and $*$ is a $t-$norm satisfying the following condition: $F_{x,y}(t + s) \geq F_{x,z}(t) * F_{z,y}(s)$,for all $x, y, z \in X$ and $t, s > 0$.
Throughout this paper, $(X, \mathcal{F}, *)$ is considered to be a Menger space with condition $\lim_{t\to\infty} \mathcal{F}_{x,y}(t) = 1$ for all $x, y \in X$.

Definition 1.5. [25] Let $(X, \mathcal{F}, *)$ be a Menger spae and $*$ be a $t-$norm. Then
(1) a sequance $\{x_n\}$ in X is said to converge to a point x in X if and only if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a integer $N \in \mathbb{N}$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ for all $n \geq N$;
(2) a sequance $\{x_n\}$ in X is said to be Cauchy if for every $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $N \in \mathbb{N}$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq N$;
(3) a Menger space $(X, \mathcal{F}, *)$ is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 1.6. [27] A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to be compatible if $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 1.7. [19] A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$, but for some $t > 0$, either $\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) \neq 1$ or the limit does not exist.

Definition 1.8. [28] A pair (A, S) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ is said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 1.9. [20] Two pairs (A, S) and (B, T) of self-mappings of a Menger space $(X, \mathcal{F}, *)$ are said to satisfy the common property (E.A) if there exists two sequences $\{x_n\}, \{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$, for some $z \in X$.

Definition 1.10. [26] A pair (A, S) of self-mappings of a nonempry set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Az = Sz$ for some $z \in X$, then $ASz = SAz$.

Remark 1.1. a) From Definition 1.8. it is easy to see that any two noncompatible self-mappings of $(X, \mathcal{F}, *)$ satisfy property (E.A) but the reverse need not to be true.
b) If self-mappings A and S of a Menger space $(X, \mathcal{F}, *)$ are compatible then they are weakly compatible but the reverse need not be true.
c) It is noticed that the notion of weak compatibility and the (E.A) property are independent to each other.

2 Main results

Let Φ is a set of all increasing and continuous functions $\phi : (0, 1] \to (0, 1]$, such that $\phi(t) > t$ for every $t \in (0, 1)$. In the following we need this result.

Theorem 2.1. *Let A, B, S and T be self-mappings of a Menger space $(X, \mathcal{F}, *)$, where $*$ is a continuous $t-$norm. Suppose that*
(1) $A(X) \subset T(X)$ or $B(X) \subset S(X)$,
(2) the pair (A, S) or (B, T) satisfies property (E.A),
(3) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges or $A(x_n)$ converges for every sequence$\{x_n\}$ in X whenever $S(x_n)$ converges,
(4) there exist $\phi \in \Phi$ and $1 \leq k < 2$ such that

$$F_{Ax,By}(t) \geq \phi\left(\min\left\{\begin{array}{c} F_{Sx,Ty}(t), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t} \min\{F_{Sx,Ax}(t_1), F_{Ty,By}(t_2)\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \max\{F_{Sx,By}(t_3), F_{Ty,Ax}(t_4)\} \end{array}\right\}\right) \tag{2.1}$$

holds for all $x, y \in X$, $t > 0$. Then the pairs (A, S) and (B, T) share the common property (E.A).

Proof. ► Suppose the pair (A, S) satisfies property (E.A). Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{2.2}$$

for some $z \in X$. Since $A(X) \subset T(X)$, hence for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z. \tag{2.3}$$

Thus in all, we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. By (3), the sequence $\{By_n\}$ converges and in all we need to show that $By_n \to z$ as $n \to \infty$. Let $By_n \to l$ for $t > 0$ as $n \to \infty$. Then, it is enough to show that $z = l$. Suppose that $z \neq l$, then there exists $t_0 > 0$ such that

$$F_{z,l}\left(\frac{2}{k}t_0\right) > F_{z,l}(t_0). \tag{2.4}$$

In order to establish the claim embodied in (2.4), let us assume that (2.4) does not hold. Then we have $F_{z,l}\left(\frac{2}{k}t\right) \leq F_{z,l}(t)$ for all $t > 0$. Repeatedly using this equality, we obtain

$$F_{z,l}(t) \geq F_{z,l}(\frac{2}{k}t) \geq \cdots \geq F_{z,l}\left(\left(\frac{2}{k}\right)^n t\right) \to 1,$$

as $n \to \infty$. This shows that $F_{z,l}(t) = 1$ for all $t > 0$, which contradicts $z \neq l$ and hence (2.4) is proved. Using inequality (2.1), with $x = x_n,\ y = y_n$, we get

$$\begin{aligned} F_{Ax_n,By_n}(t_0) &\geq \phi\left(\min\left\{\begin{array}{c} F_{Sx_n,Ty_n}(t_0), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Sx_n,Ax_n}(t_1), F_{Ty_n,By_n}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Sx_n,By_n}(t_3), F_{Ty_n,Ax_n}(t_4)\right\} \end{array}\right\}\right) \\ &\geq \phi\left(\min\left\{\begin{array}{c} F_{Sx_n,Ty_n}(t_0), \\ \min\left\{F_{Sx_n,Ax_n}(\epsilon), F_{Ty_n,By_n}\left(\frac{2}{k}t_0-\epsilon\right)\right\}, \\ \max\left\{F_{Sx_n,By_n}\left(\frac{2}{k}t_0-\epsilon\right), F_{Ty_n,Ax_n}(\epsilon)\right\} \end{array}\right\}\right), \end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $n \to \infty$, it follows that

$$\begin{aligned} F_{z,l}(t_0) &\geq \phi\left(\min\left\{\begin{array}{c} F_{z,z}(t_0), \\ \min\left\{F_{z,z}(\epsilon), F_{z,l}\left(\frac{2}{k}t_0-\epsilon\right)\right\}, \\ \max\left\{F_{z,l}\left(\frac{2}{k}t_0-\epsilon\right), F_{z,z}(\epsilon)\right\} \end{array}\right\}\right) \\ &= \phi\left(F_{z,l}\left(\frac{2}{k}t_0-\epsilon\right)\right) > F_{z,l}\left(\frac{2}{k}t_0-\epsilon\right), \end{aligned}$$

as $\epsilon \to 0$, we have $F_{z,l}(t_0) \geq F_{z,l}\left(\frac{2}{k}t_0\right)$, which is contradicts (2.4). Therefore, $z = l$. Hence the pairs (A, S) and (B, S) share the common property (E.A). ◄

Now we prove a common fixed point theorem for two pairs of mappings in Menger space.

Theorem 2.2. *Let A, B, S and T be self-mappings of a Menger space $(X, \mathcal{F}, *)$, where $*$ is a continuous $t-$norm satisfying inequality (2.1) of Lemma 2.1. Suppose that*
(1) the pairs (A, S) and (B, T) share the common property (E.A),
(2) $S(X)$ and $T(X)$ are closed subsets of X.
Then the pairs (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided both pairs (A, S) and (B, T) are weakly compatible.

Proof. ► Since the pairs (A, S) and (B, T) share the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z, \tag{2.5}$$

for some $z \in X$. Since $S(X)$ is a a closed subset of X, hence $\lim\limits_{n\to\infty} Sx_n = z \in S(X)$. Therefore, there exists a point $u \in X$ such that $Su = z$. Now we assert that $Au = Su$. Suppose that $Au \neq Su$, then there exists $t_0 > 0$ such that

$$F_{Au,Su}\left(\frac{2}{k}t_0\right) > F_{Au,Su}(t_0). \tag{2.6}$$

In order to establish the claim embodied in (2.6), let us assume that (2.6) does not hold. Then we have $F_{Au,Su}\left(\frac{2}{k}t\right) \leq F_{Au,Su}(t)$ for all $t > 0$. Repeatedly using this equality, we obtain

$$F_{Au,Su}(t) \geq F_{Au,Su}(\frac{2}{k}t) \geq \cdots \geq F_{Au,Su}\left(\left(\frac{2}{k}\right)^n t\right) \to 1,$$

as $n \to \infty$. This shows that $F_{Au,Su}(t) = 1$ for all $t > 0$, which contradicts $Au \neq Su$ and hence (2.6) is proved. Using inequality (2.1), with $x = u$, $y = y_n$, we get

$$\begin{aligned} F_{Au,By_n}(t_0) &\geq \phi\left(\min\left\{\begin{array}{c} F_{Su,Ty_n}(t_0), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Su,Au}(t_1), F_{Ty_n,By_n}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Su,By_n}(t_3), F_{Ty_n,Au}(t_4)\right\} \end{array}\right\}\right) \\ &\geq \phi\left(\min\left\{\begin{array}{c} F_{z,Ty_n}(t_0), \\ \min\left\{F_{z,Au}\left(\frac{2}{k}t_0-\epsilon\right), F_{By_n,Ty_n}(\epsilon)\right\}, \\ \max\left\{F_{z,By_n}(\epsilon), F_{Ty_n,z}\left(\frac{2}{k}t_0-\epsilon\right)\right\} \end{array}\right\}\right), \end{aligned}$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $n \to \infty$, it follows that

$$F_{Au,z}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{z,z}(t_0), \\ \min\left\{F_{z,Au}\left(\frac{2}{k}t_0 - \epsilon\right), F_{z,z}(\epsilon)\right\}, \\ \max\left\{F_{z,z}(\epsilon), F_{z,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\} \end{array}\right\}\right)$$
$$= \phi\left(F_{z,Au}\left(\frac{2}{k}t_0 - \epsilon\right)\right) > F_{Au,z}\left(\frac{2}{k}t_0 - \epsilon\right),$$

as $\epsilon \to 0$, we have $F_{Au,z}(t_0) \geq F_{Au,z}\left(\frac{2}{k}t_0\right)$, which contradicts (2.6). Therefore, $Au = Su = z$ and hence u is a coincidence point of (A, S).
If $T(X)$ is a closed subset of X, there exists a point $v \in X$ such that $Tv = z$. Now we assert that $Bv = Tv = z$. Suppose that $Bv \neq Tv$. Then there exists $t_0 > 0$ such that

$$F_{Bv,Tv}\left(\frac{2}{k}t_0\right) > F_{Bv,Tv}(t_0). \tag{2.7}$$

To support the claim, let assume that (2.7) does not hold. Then we have $F_{Bv,Tv}\left(\frac{2}{k}t\right) \leq F_{Bv,Tv}(t)$ for all $t > 0$. Repeatedly using this equality, we obtain

$$F_{Bv,Tv}(t) \geq F_{Bv,Tv}(\frac{2}{k}t) \geq \cdots \geq F_{Bv,Tv}\left(\left(\frac{2}{k}\right)^n t\right) \to 1,$$

as $n \to \infty$. This shows that $F_{Bv,Tv}(t) = 1$ for all $t > 0$, which contradicts $Bv \neq Tv$ and hence(2.7) is proved. Using inequality (2.1), with $x = x_n,\ y = v$, we get

$$F_{Ax_n,Bv}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{Sx_n,Tv}(t_0), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Sx_n,Ax_n}(t_1), F_{Tv,Bv}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Sx_n,Bv}(t_3), F_{Tv,Ax_n}(t_4)\right\} \end{array}\right\}\right)$$
$$\geq \phi\left(\min\left\{\begin{array}{c} F_{Sx_n,z}(t_0), \\ \min\left\{F_{Sx_n,Ax_n}(\epsilon), F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}, \\ \max\left\{F_{Sx_n,Bv}\left(\frac{2}{k}t_0 - \epsilon\right), F_{z,Ax_n}(\epsilon)\right\} \end{array}\right\}\right),$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $n \to \infty$, it follows that

$$F_{z,Bv}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{z,z}(t_0), \\ \min\left\{F_{z,z}(\epsilon), F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}, \\ \max\left\{F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right), F_{z,z}(\epsilon)\right\} \end{array}\right\}\right)$$
$$= \phi\left(F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right)\right) > F_{z,Bv}\left(\frac{2}{k}t_0 - \epsilon\right),$$

as $\epsilon \to 0$, we have $F_{z,Bv}(t_0) \geq F_{z,Bv}\left(\frac{2}{k}t_0\right)$, which contradicts (2.7). Therefore, $Bv = Tv = z$ and hence v is a coincidence point of (B, T)
Since the pair (A, S) is weakly compatible, therefore $Az = ASu = SAu = Sz$. Now we assert that z is a common fixed point of (A, S). If $z \neq Az$, then on using inequality (2.1), with $x = z, y = v$ we get for some $t_0 > 0$

$$F_{Az,Bv}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{Sz,Tv}(t_0), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Sz,Az}(t_1), F_{Tv,Bv}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Sz,Bv}(t_3), F_{Tv,Az}(t_4)\right\} \end{array}\right\}\right),$$

$$F_{Az,z}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{Az,z}(t_0), \\ \min\left\{F_{Az,Az}(\epsilon), F_{z,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}, \\ \max\left\{F_{Az,z}(\epsilon), F_{z,Az}\left(\frac{2}{k}t_0 - \epsilon\right)\right\} \end{array}\right\}\right),$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\epsilon \to 0$, we have

$$\begin{aligned} F_{Az,z}(t_0) &\geq \phi\left(\min\left\{F_{Az,z}(t_0), F_{z,Az}\left(\frac{2}{k}t_0\right)\right\}\right) \\ &= \phi\left(F_{Az,z}(t_0)\right) > F_{Az,z}(t_0), \end{aligned}$$

which is a contradiction. Hence, $Az = Sz = z$, i.e. z is a common fixed point of (A, S). Also, the pair (B, T) is weakly compatible, therefore $Bz = BTv = TBv = Tz$. Now we show that z is also a common fixed point of (B, T). If $z \neq Bz$, then on using inequality (2.1), with $x = u, y = z$ we get for some $t_0 > 0$

$$F_{Au,Bz}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{Su,Tz}(t_0), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t_0} \min\left\{F_{Su,Au}(t_1), F_{Tz,Bz}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t_0} \max\left\{F_{Su,Bz}(t_3), F_{Tz,Au}(t_4)\right\} \end{array}\right\}\right),$$

$$F_{z,Bz}(t_0) \geq \phi\left(\min\left\{\begin{array}{c} F_{z,Bz}(t_0), \\ \min\left\{F_{z,z}(\epsilon), F_{Bz,Bz}\left(\frac{2}{k}t_0 - \epsilon\right)\right\}, \\ \max\left\{F_{z,Bz}(\epsilon), F_{Bz,z}\left(\frac{2}{k}t_0 - \epsilon\right)\right\} \end{array}\right\}\right),$$

for all $\epsilon \in \left(0, \frac{2}{k}t_0\right)$. As $\epsilon \to 0$, we have

$$\begin{aligned} F_{z,Bz}(t_0) &\geq \phi\left(\min\left\{F_{z,Bz}(t_0), F_{Bz,z}\left(\frac{2}{k}t_0\right)\right\}\right) \\ &= \phi\left(F_{z,Bz}(t_0)\right) > F_{z,Bz}(t_0), \end{aligned}$$

which is a contradiction. Hence, $Bz = Tz = z$, i.e. z is a common fixed point of (B,T). The uniqueness of common fixed point is an easy consequence of inequality (2.1). ◄

The following example illustrates Theorem 2.1.

Example 2.1. Let $(X,\mathcal{F},*)$ be a Menger spaces , where $X = [2,19]$, with continuous t–norm, $*$ is defined by $a*b = ab$ for all $a,b \in [0,1]$ and for all $x,y \in X$ let $F_{x,y}(t) = \left(\frac{t}{t+1}\right)^{|x-y|}$. The function $\phi : (0,1] \to (0,1]$ is defined as $t^{\frac{1}{2}}$. Define the self-mappings A, B, S and T by

$$A(x) = \begin{cases} 2, \text{ if } x \in \{2\} \cup (3,19]; \\ 3, \text{ if } x \in (2,3], \end{cases} \qquad B(x) = \begin{cases} 2, \text{ if } x \in \{2\} \cup (3,19]; \\ \frac{5}{2}, \text{ if } x \in (2,3], \end{cases}$$

$$S(x) = \begin{cases} 2, \text{ if } x = 2; \\ 10, \text{ if } x \in (2,3]; \\ \frac{x+77}{40}, \text{ if } x \in (3,19], \end{cases} \qquad T(x) = \begin{cases} 2, \text{ if } x = 2; \\ 13, \text{ if } x \in (2,3]; \\ 14, \text{ if } x = 3; \\ \frac{x+77}{40}, \text{ if } x \in (3,19]. \end{cases}$$

We take $\{x_n\} = \left\{3 + \frac{1}{n}\right\}$, $\{y_n\} = \{2\}$,or $\{x_n\} = \{2\}$, $\{y_n\} = \left\{3 + \frac{1}{n}\right\}$. Hence, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2 \in X.$$

The both pairs (A,S) and (B,T) satisfy the common property (E.A). It is noted that $A(X) = \{2,3\} \subsetneq [2,2.4] \cup [13,14] = T(X)$ and $B(X) = \left\{2, \frac{5}{2}\right\} \subsetneq [2,2.4] \cup \{10\} = S(X)$. Also, $S(X)$ and $T(X)$ are closed subsets of X. All the conditions of Theorem 2.1 are satisfied and 2 is a unique common fixed point of the pairs (A,S) and (B,T) and all mappings are discontinuous at their unique common fixed point 2.

Theorem 2.3. *The conclusion of Theorem 2.1 remains true if the condition* (2) *of Theorem 2.1 is replaced by the following*

(2′) $\overline{A(X)} \subset T(X)$ *and* $\overline{B(X)} \subset S(X)$, *where* $\overline{A(X)}$ *is the closure range of* A *and* $\overline{B(X)}$ *is the closure range of* B.

Proof. ► Since the pairs (A,S) and (B,T) satisfy the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$, for some $z \in X$. Then, since $z \in \overline{A(X)}$ and $\overline{A(X)} \subset T(X)$ there exists a point $v \in X$ such that $z = Tv$. By the proof of Theorem 2.1, it can be show that the pair (B,T) has a coincidence point, for example v, i.e. $Bv = Tv$. Snce $z \in \overline{B(X)}$ and $\overline{B(X)} \subset S(X)$ there exists a point $u \in X$ such that $z = Su$. Similarly it can be also prove that the pair (A,S) has a coincidence point, call it u, i.e. $Au = Su$. The rest of the proof is on the lines of the proof of Theorem 2.1.◄

Corollary 2.1. *The conclusion of Theorems 2.1-2.2 remain true if the condition* (2) *of Theorem 2.1 and condition* (2′) *of Theorem 2.2 are replaced by the following:* ($2''$) $A(X)$ *and* $B(X)$ *are closed subsets of* X *provided* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$.

Theorem 2.4. *Let* $(X, \mathcal{F}, *)$ *be a Menger spaces, where* $*$ *is continuous t–norm. Let* A, B, S *and* T *be mappings from* X *into itself and satisfying the condition* $(1) - (4)$ *of Lema 2.1. Suppose that*
(5) $S(X)$ *or* $T(X)$ *is a closed subset of* X.
Then the pairs (A, S) *and* (B, T) *have a coincidence point each. Moreover,* A, B, S *and* T *have a unique common fixed point provided both pairs* (A, S) *and* (B, T) *are weakly compatible.*

Proof. ► In view of Lemma 2.1, the pairs (A, S) and (B, T) share the common property (E.A), i.e. there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$, for some $z \in X$.

If $S(X)$ is a closed subset of X, then on the lines of Theorem 2.1, it can be show that the pair (A, S) has coincidence point, say u, i.e. $Au = Su = z$. Since $A(X) \subset T(X)$ and $Au \in A(X)$, there exists a point $v \in X$ such that $Au = Tv$. The rest of the proof runs along the lines of the proof of Theorem 1.◄

Example 2.2. In setting of Example 2.1. replace the self-mappings A, B, S and T by

$$A(x) = \begin{cases} 2, \text{ if } x \in \{2\} \cup (3, 19]; \\ 3, \text{ if } x \in (2, 3], \end{cases} \qquad B(x) \begin{cases} 2, \text{ if } x \in \{2\} \cup (3, 19]; \\ 4, \text{ if } x \in (2, 3], \end{cases}$$

$$S(x) = \begin{cases} 2, \text{ if } x = 2; \\ 14, \text{ if } x \in (2, 3]; \\ \frac{x+1}{2}, \text{ if } x \in (3, 19], \end{cases} \qquad T(x) = \begin{cases} 2, \text{ if } x = 2; \\ 11 + x, \text{ if } x \in (2, 3]; \\ \frac{x+1}{2}, \text{ if } x \in (3, 19]. \end{cases}$$

It is noted that $A(X) = \{2, 3\} \subset [2, 10] \cup (13, 14] = T(X)$ and $B(X) = \{2, 4\} \subset [2, 10] \cup \{14\} = S(X)$. The pairs (A, S) and (B, T) commute at their common coincidence point 2. All the conditions of Theorems 2.2-2.3 and Corollary 2.1 are satisfied and 2 is a unique common fixed point of A, B, S and T. Here, it may be pointed out that Theorem 2.1 is not applicable to this example as $S(X)$ is not closed subset of X. Also, notice that some mappings in this example are discontinuous at their unique common fixed point 2.

By choosing A, B, S and T suitably, we can drive a multitude of common fixed point theorems for a pair or triod of self-mappings. If we take $A = B$ and $S = T$ in Theorem 2.1 then we get the following result

Corollary 2.2. *Let* $(X, \mathcal{F}, *)$ *be a Menger spaces, where* $*$ *is continuous t–norm. Let* A *and* S *be mappings from* X *into itself and satisfying the following conditions:*
(1) *The pair* (A, S) *shares property (E.A),*
(2) $S(X)$ *is a closed subset of* X,

(3) *there exist* $\phi \in \Phi$ *and* $1 \leq k < 2$ *such that*

$$F_{Ax,Ay}(t) \geq \phi\left(\min\left\{\begin{array}{c} F_{Sx,Sy}(t), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t} \min\left\{F_{Sx,Ax}(t_1), F_{Sy,Ay}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \max\left\{F_{Sx,Ay}(t_3), F_{Sy,Ax}(t_4)\right\} \end{array}\right\}\right), \qquad (2.8)$$

holds for all $x, y \in X$ *and* $t > 0$. *Then the pair* (A, S) *has a coincidence point. Moreover, A and S have a unique common fixed point provided the pair* (A, S) *is weakly compatible.*

At the end, the next theorem is valid for six self-mappings in Menger spaces.

Theorem 2.5. *Let* $(X, \mathcal{F}, *)$ *be a Menger spaces, where* $*$ *is continuous* t*–norm. Let* A, B, R, S, H *and* T *be mappings from* X *into itself and satisfying the following conditions:*
(1) *The pairs* (A, SR) *and* (B, TH) *shares the common property (E.A),*
(2) $SR(X)$ *and* $TH(X)$ *are closed subsets of* X,
(3) *there exist* $\phi \in \Phi$ *and* $1 \leq k < 2$ *such that*

$$F_{Ax,By}(t) \geq \phi\left(\min\left\{\begin{array}{c} F_{SRx,THy}(t), \\ \sup\limits_{t_1+t_2=\frac{2}{k}t} \min\left\{F_{SRx,Ax}(t_1), F_{THy,By}(t_2)\right\}, \\ \sup\limits_{t_3+t_4=\frac{2}{k}t} \max\left\{F_{SRx,By}(t_3), F_{THy,Ax}(t_4)\right\} \end{array}\right\}\right), \qquad (2.8)$$

holds for all $x, y \in X$ *and* $t > 0$. *Then the pairs* (A, SR) *and* (B, TH) *have a coincidence point each. Moreover,* A, B, R, S, H *and* T *have a unique common fixed point provided the pairs* (A, SR) *and* (B, TH) *commute pairwise (i.e.* $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ *and* $TH = HT$*).*

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Optimal Process Calibration for Some Examples of Non-symmetric Loss Functions

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Abstract

The process calibration is one of the methods in statistical process control for improving the quality of the products. We will introduce basic notions of statistical process control and give a review on loss functions used in this area. This paper is concerned with the problem of process calibration for some examples of non-symmetric loss functions. We suggest an optimal calibration policy which can be implemented using our programs written in statistical software R.

1 Introduction

Statistical quality control is collection of statistical methods and other problem-solving techniques for improving the quality of the products used by our society. These products consist of manufactured goods such as automobiles, computers and clothing, as well as services provided by telephone companies, hotels, banks, etc. Quality improvement methods can be applied to any area within a company - from engineering design and manufacturing to customer service [3].

Quality characteristic represents key feature of the product that can be measured. It is a random variable that can be continuous (length, diameter, weight, thickness of a product) or discrete (the number of errors or mistakes made in completing a loan application, the number of medical errors made in a hospital). Desired value of quality characteristic is called the nominal or target value which is usually bounded by a interval of values called specification limits. It is considered that specification limits will be sufficiently close to the target value so as to not impact the function or performance of the product if the value of quality characteristic is in that range. The largest allowable value for a quality characteristic is called the upper specification limit (USL), and the smallest allowable value for a quality characteristic is called the lower specification limit (LSL). Some quality characteristics have specification limits on only one side of the target. A product is called nonconforming if the value of quality characteristic lies outside interval of specification limits [LSL, USL]. For example, diameter of assembly component cannot be too large or it will not fit, nor can it be too small, resulting in a loose fit, causing vibration, wear, and early failure of the assembly. The loss function gives us a way to calculate the "quality loss" obtained for producing nonconforming products.

In this paper we consider the problem of the process calibration, i.e. how to set-up a manufacturing process in order to meet specification limits (minimize expected loss). In Section 2, we will introduce traditional method of calibration which corresponds to the symmetric loss function. However, quite often the lower and upper specification limits cannot be treated in the same way. For example, it may happen that although one of the specification limits can be exceeded, we obtain a nonconforming product that could be reworked (improved or corrected) which requires some costs. Three models of non-symmetric loss functions and their implementation in statistical software R are considered, respectively, in Sections 3 and 4. Appendix consists of R code. Finally, conclusions are given in Section 5.

2 Symmetric loss function

Let X denote a quality characteristic of the product. We will assume that X is normally distributed with mean μ and standard deviation σ, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$. We will suppose that the standard deviation σ is known.

Usual practice is to set-up the mean at the middle distance between the specification limits, i.e. $\mu_0 = \frac{LSL+USL}{2}$ and this is legitimate only if loss does not depend on which specification limit - the lower or upper - is exceeded. In that case the loss function L(x) is symmetrical,

$$L(x) = \begin{cases} w, & x < LSL \\ 0, & LSL \leq x \leq USL \\ w, & x > USL \end{cases}$$

where $w > 0$ (see Figure 1).

This is very simple method of process calibration. However, very often the assumption of the symmetrical loss function is not fulfilled in practice. Let us consider three following examples.

Example 2.1. The quality characteristic under study is the inner diameter of a hole. Undersized holes can be reworked at extra costs. But the reduction of the over-sized hole would be either impossible or would require much higher costs.

Figure 1: Symmetric loss function

Example 2.2. The quality characteristic under study is the outside diameter of the milled item. Over-sized items can be re-grounded at additional costs. But the undersized item could be sold only for scrap.

Example 2.3. The quality characteristic under study is the blood glucose level in diabetes patients. If measured value of blood sugar is too low or too high, i.e. below or above allowed limits, then the blood sugar can be stabilized with some extra costs (injection of insulin, food, tablets, etc.). These costs does not have to be the same for too low and too high level of blood sugar.

These examples show that in situations described above, instead of the symmetrical loss function, a non-symmetric loss function should be used.

3 Non-symmetric loss function

To find optimal calibration policy for situations as described in the examples given above we will consider three models of loss functions. First model was proposed by Grzegorzewski and Mrowka [2] and other two models are suggested by the authors.

3.1 First model

The first model of the loss function is given by the formula

$$L(x) = \begin{cases} w, & x < LSL \\ 0, & LSL \geq x \leq USL \\ z\frac{x-USL}{ULR-USL}, & USL < x \leq ULR \\ z, & x > ULR \end{cases}$$

where $w > z$ and upper limit for rework $ULR > USL$ (see Fig.2).

This model could be used in cases when over-sized product items can be reworked with some extra costs (Example 2). The expected loss function is equal to

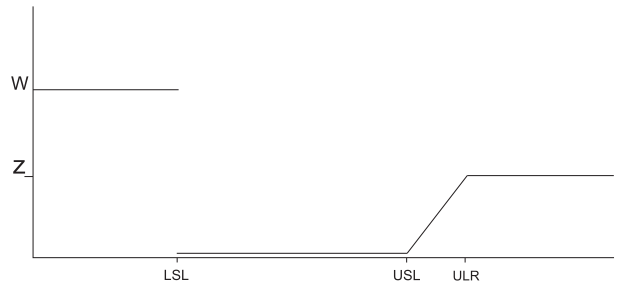

Figure 2: The first model of the loss function

$$EL(x) = wP(x < LSL) + zP(x > ULR) + z\int_{USL}^{ULR} \frac{x - USL}{ULR - USL} f(x)\mathrm{d}x$$

where $f(x)$ is normal density function of quality characteristic X. After some transformations we get

$$\begin{aligned} EL(x) &= w\phi(\frac{LSL-\mu}{\sigma}) + z + z(\frac{\mu - USL}{ULR - US} - 1)\Phi(\frac{ULR-\mu}{\sigma}) + \\ &+ z\frac{\mu - USL}{ULR - USL}\Phi\left(\frac{USL-\mu}{\sigma}\right) - \\ &- \sigma\frac{z}{ULR-USL}\left(\phi\left(\frac{ULR-\mu}{\sigma}\right) - \phi\left(\frac{USL-\mu}{\sigma}\right)\right), \end{aligned}$$

where Φ and ϕ are, respectively, distribution function and density function of standard normal distribution $\mathcal{N}(0,1)$. We consider the expected loss as a function of a process mean μ, $EL = EL(\mu)$ and we are searching for such μ in which minimal expected loss occurs. Thus we need

$$\begin{aligned} &\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu} = \\ = &\frac{z}{ULR-USL}\left[\Phi\left(\frac{ULR-\mu}{\sigma}\right) - \Phi\left(\frac{USL-\mu}{\sigma}\right)\right] - \frac{w}{\sigma}\phi\left(\frac{LSL-\mu}{\sigma}\right) = 0. \end{aligned}$$

This equation depends on μ and also LSL, USL, ULR, w, z, σ. To simplify it, we will use new parameters:

$$\begin{aligned} a &= \frac{ULR-USL}{\sigma}, \\ b &= \frac{USL-LSL}{\sigma}, \\ K &= \frac{w}{z}. \end{aligned}$$

Optimal calibration is given by

$$\mu = USL - \Delta\sigma, \tag{1}$$

where Δ is correction factor.

Previous equation now becomes

$$\frac{1}{a}\left[\Phi(a+\Delta) - \Phi(\Delta)\right] - K\phi(-b+\Delta) = 0. \tag{2}$$

For cases when undersized items are reworked with some extra costs (Example 1), we could consider loss function which reflects first model loss function.

Following two models could be appropriate in the cases when there is possibility of rework for both undersized and oversized items (Example 3).

3.2 Second model

Loss function is given by the formula

$$L(x) = \begin{cases} 4w, & x < LLR \\ 3w, & LLR \leq x < LSL \\ 0, & LSL \leq x < USL \\ w, & USL \leq x \leq ULR \\ 2w, & x > ULR \end{cases},$$

where $w > 0$ (see Fig.3).

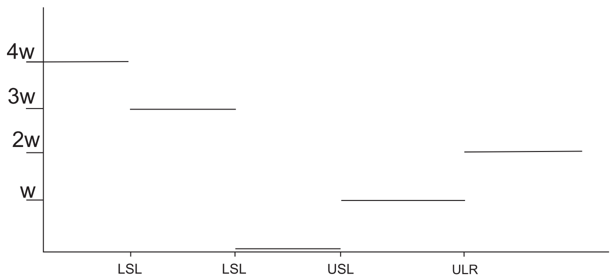

Figure 3: The second model of the loss function

In this situation the expected loss is equal to

$$EL(x) = wP(X < LLR) + 3wP(X \leq LSL) - wP(X \leq ULR) - wP(X \leq USL).$$

After some transformations we get

$$EL(x) = w\Phi\left(\frac{LLR-\mu}{\sigma}\right) + 3w\Phi\left(\frac{LSL-\mu}{\sigma}\right) - w\Phi\left(\frac{ULR-\mu}{\sigma}\right) - w\Phi\left(\frac{USL-\mu}{\sigma}\right)$$

and, further $\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu}$ is equal to

$$\frac{w}{\sigma}\left[\phi\left(\frac{LLR-\mu}{\sigma}\right)+3\phi\left(\frac{LSL-\mu}{\sigma}\right)-\phi\left(\frac{ULR-\mu}{\sigma}\right)-\phi\left(\frac{USL-\mu}{\sigma}\right)\right]=0.$$

After dividing this equation with $-\frac{w}{\sigma}$ and using parameters $a=\frac{ULR-USL}{\sigma}=\frac{LSL-LLR}{\sigma}$, b and Δ as before, we get

$$\phi(-a-b-\Delta)+3\phi(-b+\Delta)-\phi(a+\Delta)-\phi(\Delta)=0. \tag{3}$$

3.3 Third model

This model is given by the formula

$$L(x)=\begin{cases} w_2, & x<LLR \\ w_1\frac{x-LLR}{LSL-LLR}-w_2\frac{x-LSL}{LSL-LLR}, & LLR\geq x<LSL \\ 0, & LSL\geq x\leq USL \\ z\frac{x-USL}{ULR-USL}, & USL<x\leq ULR \\ z, & x>ULR \end{cases}$$

where $w_2>w_1>z$ (see Fig.4).

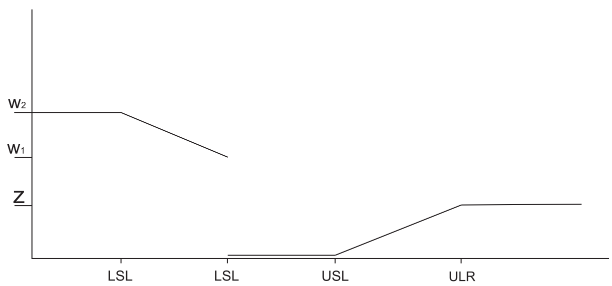

Figure 4: The third model of the loss function

In this situation the expected loss is equal to

$$\begin{aligned} EL(x) &= \left[w_2+\frac{w_2-w_1}{a\sigma}\mu+w_1\frac{LLR}{a\sigma}-w_2\frac{LSL}{a\sigma}\right]\Phi\left(\frac{LLR-\mu}{\sigma}\right)+ \\ &+ \left[\frac{w_1-w_2}{a\sigma}\mu-w_1\frac{LLR}{a\sigma}++w_2\frac{LSL}{a\sigma}\right]\Phi\left(\frac{LSL-\mu}{\sigma}\right)+ \\ &+ \left[-\frac{z}{a\sigma}\mu++z\frac{USL}{a\sigma}-z\right]\Phi\left(\frac{USL-\mu}{\sigma}\right)+ \\ &+ \left[\frac{z}{a\sigma}\mu-z\frac{USL}{a\sigma}\right]\Phi\left(\frac{ULR-\mu}{\sigma}\right)+ \\ &+ \frac{w_1-w_2}{a}\phi\left(\frac{LLR-\mu}{\sigma}\right)+\frac{w_2-w_1}{a}\phi\left(\frac{LSL-\mu}{\sigma}\right)+ \end{aligned}$$

$$+ \quad \frac{z}{a}\phi\left(\frac{USL-\mu}{\sigma}\right) - \frac{z}{a}\phi\left(\frac{ULR-\mu}{\sigma}\right),$$

where $a = \frac{ULR-USL}{\sigma} = \frac{LSL-LLR}{\sigma}$.

Further, we have

$$\begin{aligned}
\frac{\mathrm{d}EL(\mu)}{\mathrm{d}\mu} &= \frac{w_2-w_1}{a\sigma}\Phi\left(\frac{LLR-\mu}{\sigma}\right) - \frac{w_2-w_1}{a\sigma}\Phi\left(\frac{LSL-\mu}{\sigma}\right) + \\
&+ \frac{z}{a\sigma}\left[\Phi\left(\frac{ULR-\mu}{\sigma}\right) - \Phi\left(\frac{USL-\mu}{\sigma}\right)\right] - \frac{w_1}{\sigma}\phi\left(\frac{LSL-\mu}{\sigma}\right) + \\
&+ \frac{z}{\sigma}\left[\phi\left(\frac{USL-\mu}{\sigma}\right) - \phi\left(\frac{ULR-\mu}{\sigma}\right)\right] = 0.
\end{aligned}$$

Using parameters a,b, $K_1 = \frac{w_1}{z}$ and $K_2 = \frac{w_2}{z}$, we get

$$\begin{aligned}
&\frac{K_2-K_1}{a}\left[\Phi(-a-b+\Delta) - \Phi(-b+\Delta)\right] + \frac{\Phi(a+\Delta)-\Phi(\Delta)}{a} - \\
-\ &K_1\phi(-b+\Delta) + \phi(\Delta) - \phi(a+\Delta) = 0. \qquad (4)
\end{aligned}$$

4 Implementation in R

To optimally calibrate the process mean (equation (1)), we need to calculate the correction factor Δ. For three considered models, Δ is calculated as a solution of equations (2),(3) and (4), respectively. These equations depend on various parameters - equation (2) depends on tree parameters a, b and K, equation (3) on two parameters a and b and equation (4) depends on four parameters a, b, K_1 and K_2. We will solve these equations numerically, using Brent's root- finding method [1] implemented in R function *uniroot*.

Correction factor for these three models of non-symmetric loss functions can be found calling our R function *corr.factor* (code is presented in Appendix).

```
corr.factor(delta,a,b,K,K1,K2,par)
```

by specifying the values of parameters (by choosing non-zero values for parameters a, b and/or K, $K1$ and $K2$). Parameter *par* represents upper limit for correction function necessary for Brent's method to work properly (by default value of parameter $par = 3$).

Now, we want to find correction factors for considered non-symmetric loss functions for given values of parameters.

Example 4.1. Parameters of the first model of non-symmetric loss function a, b and K have following values: $a = 12$, $b = 6$ and $K = 2$. By calling function

```
corr.function(a=12,b=6,K=2)
```

we get $\Delta = 2.308$.

Example 4.2. Parameters of the second model of non-symmetric loss function a, b have following values: $a = 12$, $b = 6$. By calling function

```
corr.function(a=12,b=6)
```

we get $\Delta = 2.817$.

Example 4.3. Parameters of the third model of non-symmetric loss function a, b and $K1$ and $K2$ have following values: $a = 12$, $b = 6$, $K1 = 2$, $K2 = 3$. By calling function

```
corr.function(a=12,b=6,K1=2,K2=3)
```

we get $\Delta = 2.887$.

5 Conclusions

We have considered the problem of process calibration for three examples of non-symmetric loss function which can be used when losses caused by producing oversized or undersized items are not equal. We suggested an optimal calibration policy which can be implemented using our programs written in statistical software R.

6 Appendix

```
model1<-function(delta,a,b,K){
1/a*(pnorm(a+delta)-pnorm(delta))-K*dnorm(-b+delta)}

model2<-function(delta,a,b){
dnorm(-a-b+delta)+3*dnorm(-b+delta)-dnorm(a+delta)-dnorm(delta) }


model3<-function(delta,a,b,K1,K2){
(K2-K1)/a*(pnorm(-a-b+delta)-pnorm(-b+delta))+1/a*(pnorm(a+delta)-
pnorm(delta))-K1*dnorm(-b+delta)+dnorm(delta)-dnorm(a+delta) }


corr.factor<-function(delta,a=3/0.1,b=3,K=0,K1=0,K2=0,par=3) {
 if(K==0&&K1==0&K2==0)
   {model.2<-function(delta){model2(delta,a,b)}
    solution<-uniroot(model2,lower=0,upper=par)$root
   }

 else if(K1>0&&K2>0)
  {model.3<-function(delta){model3(delta,a,b,K1,K2)}
```

```
   solution<-uniroot(model3,lower=0,upper=par)$root
  }

 else if(K>0)
   {model.1<-function(delta){model1(delta,a,b,K)}
    solution<-uniroot(model1,lower=0,upper=par)$root
    }

solution }
```

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On Harmonic Mean Values and Weakly Mixing Transformations

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Original scientific paper

Abstract

Let (S,d) be a (extended) metric space, and let A be a subset of S. For any positive integer $k (k \geq 2)$, we define the *harmonic diameter of order* k of A, denoted by $D_k(A)$, by

$$D_k(A) := \sup\left\{ \binom{k}{2} \Big(\sum_{1 \leq i < j \leq k} [d(x_i, x_j)]^{-1} \Big)^{-1} \,|\, x_1, \ldots, x_k \in A \right\}.$$

Note that $D_2(A)$ is the (ordinary) diameter of the set A. The main result in this work is the following theorem:

In the above situation if $(S, \mathfrak{A}, P)$ is a probability space with the property that every open ball in (S,d) is P-measurable and has positive measure and if ϕ is a transformation on S that is weakly mixing with respect to P and A is P-measurable with $P(A) > 0$, then $\limsup_{n\to\infty} D_k(\phi^n(A)) = D_k(S)$.

This extends the known result by R. E. Rice ([13], Theorem 2) (motivated by some physical phenomena and offer some clarifications of these phenomena) from ordinary diameter to harmonic diameter of any finite order.

The obtained results here also extend and/or complement the previous results due to T. Erber, B. Schweizer and A. Sklar [4], H. Fatkić and M. Brkić [7], N. S. Landkof [10], E. B. Saff [14] and C. Sempi [16].

1 Introduction

In this work we shall investigate some metric properties and dispersive effects of weakly mixing (WM) transformations on general metric spaces endowed with a normalised (probability) measure; in particular, we shall study connections of WM discrete time dynamical systems with the theory of *harmonic diameters.*

Suppose $(S, \mathfrak{A}, P)$ is a probability space. As usual, a transformation $\phi : S \to S$ is called:

(i) *weakly mixing* (or *weak-mixing*) (with respect to P if ϕ is P - measurable and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |P(\phi^{-i}(A) \cap B) - P(A)P(B)| = 0 \tag{1.1}$$

for any two P - measurable subsets A, B of S;

(ii) *strongly mixing* (or *strong-mixing*) (with respect to P) if ϕ is P - measurable and

$$\lim_{n\to\infty} P(\phi^{-n}(A) \cap B) = P(A)P(B) \tag{1.2}$$

for any two P - measurable subsets A, B of S.

If $(S, \mathfrak{A}, P)$ is a probability space, and $\phi : S \to S$ is a measure-preserving transformation (with respect to P), then we say that $\Phi := (S, \mathfrak{A}, P, \phi)$ is an *abstract dynamical system.* An abstract dynamical system is often called a *dynamical system with discrete time.* We shall say that the abstract dynamical system Φ is *weakly* (resp. *strongly*) mixing if ϕ is weakly (resp. strongly) mixing (see [3, pp. 6 - 26]).

If $\phi : S \to S$, in addition to being strongly mixing on S with respect to P, is invertible, then (1.2) is equivalent to

$$\lim_{n\to\infty} P(\phi^{n}(A) \cap B) = P(A)P(B) \tag{1.3}$$

for any P - measurable subsets A, B of S.

In this work we consider a metric space/(extended metric spaces) on which a probability measure P is defined. The domain of P, a σ - algebra $\mathfrak{A}$ of subsets of S, is assumed to include all Borel sets in (S, d); in particular, therefore, all open balls in (S, d) are P - measurable.

The (ordinary) *diameter* of a subset A of S, i.e., the supremum of the set $\{d(x, y)|x, y \in A\}$, will be denoted by $\mathrm{diam}(A)$.

Example 1.1. a) *Measuring Productivity by the Number of Units Produced Within Certain Period of Time:*
Example:
Worker A makes 48 units in 8 hours;
Worker B makes 80 units in 8 hours;

Worker C makes 96 units in 8 hours;
Worker D makes 120 units in 8 hours.
If the average productivity is defined by *arithmetic mean*, then

$$x := \frac{48 + 80 + 96 + 120}{4} = \frac{344}{4} = 86 \text{ units in 8 hours.}$$

b) *Measuring Productivity by the Reciprocal Value, i.e. by the Amount of Time Needed to Produce One Unit:*
From the aforementioned example one has:
Worker A needs 8 (hours): 48 (units) = 10 (minutes / per unit);
Worker B needs 8 (hours): 80 (units) = 6 (minutes / per unit);
Worker C needs 8 (hours) : 96 (units) = 5 (minutes / per unit);
Worker D needs 8 (hours): 120 (units) = 4 (minutes / per unit).

If the average productivity, in this case, is defined by arithmetic mean one has

$$\bar{x} := \frac{10 + 6 + 5 + 4}{4} = \frac{25}{4} = 6.25 \text{ (minutes / per unit)}, \tag{1.4}$$

and if by the *harmonic mean* then

$$H := \frac{4}{\frac{1}{10} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4}} = 5.581 \text{ (minutes / per unit)}. \tag{1.5}$$

Testing formula (1.4): If every worker works 6.25 minutes, then 4 of them (based on their individual productivities) must produce 4 units in 6.25 minutes.
We have: Worker A makes 6.25 : 10 units;
Worker B makes 6.25 : 6 units;
Worker C makes 6.25 : 5 units;
Worker D makes 6.25 : 4 units.

Total: 4.479 units

Contradiction! This means that we must not employ the arithmetic mean for this type of problem, since it yields incorrect result.

Testing formula (1.5): If every worker works 5.581 minutes, the 4 of them (based on their individual productivities) must produce 4 units in 5.581 minutes. We have:

Worker A makes 5.581 : 10 units;
Worker B makes 5.581 : 6 units;
Worker C makes 5.581 : 5 units;
Worker D makes 5.581 : 4 units.

Total: 4 units

There is no contradiction. This shows that harmonic mean yields correct result for this type of problem.

$$(H := \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} \Leftrightarrow n = \sum_{i=1}^{n} \frac{H}{x_i}.)$$

Let (S, d) be a (extended) metric space, and let A be a subset of S. For any positive integer $k (k \geq 2)$, we define the *harmonic diameter of order* k of A (kth *harmonic diameter* of A), denoted by $D_k(A)$, by

$$D_k(A) := \sup \left\{ \binom{k}{2} \Big(\sum_{1 \leq i < j \leq k} [d(x_i, x_j)]^{-1} \Big)^{-1} \mid x_1, \ldots, x_k \in A \right\}. \tag{1.6}$$

Note that $D_2(A)$ is the (ordinary) diameter (diam (A)) of the set A.

The sequence $D_k(A)$ can be shown to be monotone nonincreasing (see, e.g., [5], [8], [10] and [14]), and therefore has a limit as k tends to infinity. By definition, the *harmonic transfinite diameter* (the *logarithmic capacity*) of A is

$$\tau(A)(= \text{cap}(A)) := \lim_{k \to \infty} D_k(A). \tag{1.7}$$

Note that $0 \leq \tau(A) \leq D_k(A) \leq \text{diam}(A)$, and that

$$B \subseteq A \text{ implies } \tau(B) \leq \tau(A).$$

If A consists of only finitely many points, then $\tau(A) = 0$.

The general theory of generalised diameters and transfinite diameters plays an important role in complex analysis. It is related to the logarithmic potential theory with applications to approximation theory and the Čebyšev constant (see, e.g., [5], [10] and [14]).

Example 1.2. In the classical case, when $S = \mathbf{R}^3$, d the Euclidean metric on $\mathbf{R}^3$, and A is a compact subset of S, the value $D_k(A) (k \in (\mathbf{N} \setminus 1))$ has a simple physical interpretation. In fact, if we look the distribution in A of k point sources of electric charge of size $1/k$, then the minimum potential energy of such a system of charges obtained when these charges are in the points $q_i^{(k)} (i = 1, \ldots, k)$ where the function f, defined by

$$f(x_1, \ldots, x_k) := \binom{k}{2}^{-1} \sum_{1 \leq i < j \leq k} [d(x_i, x_j)]^{-1},$$

achieves a minimum value. In addition, the value of the potential energy is

$$\frac{k-1}{2k} \cdot \frac{1}{D_k(A)}.$$

Any set of k points that attains this minimal energy is called an *equilibrium configuration* for A.

Investigations in [4] - [8], [13] and [15]) have shown, however, that many important consequences of (1.3) persist in the absence of invertibility and/or the strongly mixing property. The following results (the most useful results of these investigations for the goals of this paper) is due to R.E. Rice [13, Theorem 2] (see also [4], [6] and [16]):

Theorem 1.1. *Let (S, d) be a metric space, let $\mathfrak{A}$ be a σ - algebra of subsets of S and P a probability measure on $\mathfrak{A}$. Suppose further that every open ball in (S, d) is P - measurable and has positive measure. Let ϕ be a transformation on S that is strongly mixing with respect to P and suppose that A is P - measurable subset of S with positive measure. Then*

$$\lim_{n\to\infty} \operatorname{diam}(\phi^n(A)) = \operatorname{diam}(S). \tag{1.8}$$

Theorem 1.1. has many consequences which are of interest because of the extreme simplicity of both their mathematical and physical realizations. Among others, these consequences have great relevance in the discussion of the recurrence paradox of Statistical Mechanics (see the previous results of T. Erber, B. Schweizer and A. Sklar [4], B. Schweizer and A. Sklar [15, pp. 181-190 and (in Dover Edition) 295-297] and H. Fatkić, S. Sekulović, and Hana Fatkić [8]).

To paraphrase from [4]: Perhaps the most significant implications of Theorem 1.1 arise in connection with a question that goes back to the very origins of ergodic theory, the famous controversy between E. Zermelo and L. Boltzmann about the relevance of the conclusion of the Poincaré recurrence theorem (the Poincaré recurrence theorem states that certain systems will, after a sufficiently long but finite time, return to a state very close to the initial state).

It is therefore interesting to investigate how the conclusions of Theorem 1.1 must be modified when strongly mixing is replaced by weakly mixing and/or the ordinary diameter is replaced by the harmonic diameter of any finite order.

2 Main results

The main result in this work is the following:

In the above situation if $(S, \mathfrak{A}, P)$ is a probability space with the property that every open ball in (S, d) is P-measurable and has positive measure and if ϕ is a transformation on S that is weakly mixing with respect to P and $A4$ is P - measurable with $P(A) > 0$, then $\limsup_{n\to\infty} D_k(\phi^n(A)) = D_k(S)$ This extends Theorem 1.1 (motivated by some physical phenomena and offer some clarifications of these phenomena) from strongly mixing to weakly mixing and/or from ordinary diameter to harmonic diameter of any finite order.

In this direction we have:

Theorem 2.1. *Let (S, d) be a metric space, let $\mathfrak{A}$ be a σ - algebra of subsets of S and P a probability measure on $\mathfrak{A}$. Suppose further that every open ball in (S, d) is P - measurable and has positive measure. Let ϕ be a transformation on S that* *is weakly mixing with respect to P and suppose that A is P - measurable subset of S with positive measure. Then*

$$\limsup_{n\to\infty} D_k(\phi^n(A)) = D_k(S), \tag{2.1}$$

where D_k is the harmonic diameter of order k given by (1.6).

Proof. Throughout this proof we will denote $D_2(S)$, the diameter of S, by D, and $\sum_{1\le i<j\le k} a_{ij}$ by $\sum_k a_{ij}$.

Let k be an arbitrary positive integer greater than 1. Since D_k is a monotone nondecreasing set function it is clear that (2.1) holds when $D_k(S) = 0$.

Suppose next that $0 < D_k(S) < +\infty$, whence also $0 < D < +\infty$, and let $\varepsilon > 0$ be given. Then, by (1.6), there exist points $x_1, x_2, \ldots, x_k$ in S such that

$$\binom{k}{2}\left\{\sum_k [d(x_i, x_j)]^{-1}\right\}^{-1} > D_k(S) - \frac{\varepsilon}{2}. \tag{2.2}$$

Let m be a positive integer and let $B_1, B_2, \ldots, B_k$ be open balls of radius $l/2m$ centered at $x_1, x_2, \ldots, x_k$, respectively.

Since ϕ is weakly mixing, we have (see [6])

$$\lim_{n\to\infty, n\notin J_i} P(A\cap\phi^{-1}(B_i)) = P(A)P(B_i), \quad (i = 1, 2, \ldots, k), \tag{2.3}$$

where $J_1, J_2, \ldots, J_k$ are subsets of the set of non-negative integers $\mathbf{N}_0 := \{0, 1, 2, 3, \ldots\}$ of density zero, i.e., for $i = 1, 2, \ldots, k$, J_i is a subset of $\mathbf{N}_0$ such that number of elements in $J_i \cap \{0, 1, 2, 3, \ldots, n-1\}$ divided by n tends to 0 as $n \to \infty$. A finite union of such sets of density zero has an infinite complement in $\mathbf{N}_0$.

Set $J = \bigcup_{i=1}^k J_i$. Since $P(A)P(B_i) > 0$ for all $i \in 1, \ldots, k$, it follows from (2.3) that there is a positive integer $n_0 (n_0 = n_0(k))$ such that

$$P(A\cap\phi^{-n}(B_i)) > 0,$$

for $i = 1, 2, \ldots, k$ and all $n \ge n_0, n \in J$, whence

$$A\cap\phi^{-n}(B_i) \ne \emptyset \tag{2.4}$$

for $i = 1, 2, \ldots, k$ and all $n \ge n_0, n \in J$. Since

$$\phi^{-n}(\phi^n(A)\cap B_i) = \phi^{-n}(\phi^n(A)) \cap \phi^{-n}(B_i) \supseteq A\cap\phi^{-n}(B_i) \tag{2.5}$$

for all $i = 1, 2, \ldots, k$ and every $n \in \mathbf{N}, n > n_0$, it follows, by (2.3), that $\phi^{-n}(\phi^n(A)\cap B_i) \ne \emptyset$ and therefore

$$\phi^n(A)\cap B_i \ne \emptyset \tag{2.6}$$

for $i = 1, 2, \ldots, k$ and all $n \ge n_0, n \in J$, where J is a subset of the set of non-negative integers $\mathbf{N}_0 := \{0, 1, 2, 3, \ldots\}$ of density zero.

Given $n \geq n_0$ and $n \notin J$, for each $i = 1, 2, \ldots, k$, choose a point y_i in $\phi^n(A) \cap B_i$. Then, for all $i, j = 1, \ldots, k$,

$$d(x_i, x_j) \leq d(x_i, y_i) + d(y_i, y_j) + d(y_j, x_j) < d(y_i, y_j) + \frac{1}{m},$$

whence, by (2.2), we have

$$D_k(S) - \frac{\varepsilon}{2} < \binom{k}{2}\Big\{\sum_k [d(x_i, x_j)]^{-1}\Big\}^{-1} < \binom{k}{2}\Big\{\sum_k \Big[d(y_i, y_j) + \frac{1}{m}\Big]^{-1}\Big\}^{-1}.$$

Since

$$\binom{k}{2}\Big\{\sum_k [d(y_i, y_j)]^{-1}\Big\}^{-1} \leq D_k(\phi^n(A))$$

and

$$\begin{aligned}\binom{k}{2}\Big\{\sum_k \Big[d(y_i, y_j) + \frac{1}{m}\Big]^{-1}\Big\}^{-1} &= \binom{k}{2}\Big\{\sum_k [d(y_i, y_j)]^{-1}\Big\}^{-1} \\ &+ \binom{k}{2}\Big\{\sum_k \Big[d(y_i, y_j) + \frac{1}{m}\Big]^{-1} - \Big\{\sum_k [d(y_i, y_j)]^{-1}\Big\}^{-1}\Big\}^{-1} \\ &\leq \binom{k}{2}\Big\{\sum_k [d(y_i, y_j)]^{-1}\Big\}^{-1} + \Delta(k, m).\end{aligned}$$

where $\Delta(k, m)$ is the positive value such that $\lim_{m\to\infty} \Delta(k, m) = 0$, we have

$$D_k(\phi^n(A)) + \Delta(k, m) > D_k(S) - \frac{\varepsilon}{2}.$$

Hence, and from the fact that there is a positive integer m_0 such that $\Delta(k, m) < \varepsilon/2$ (k fixed) for all $m \geq m_0$, it follows that

$$D_k(\phi^n(A)) > D_k(S) - \varepsilon$$

holds for every $n \geq n_0, n \in J$, i. e., for infinitely many positive integers n, and every $\varepsilon > 0$. But, clearly, $D_k(\phi^n(A)) \leq D_k(S)$ for every positive integer n, whence we obtain (2.1).

Lastly, the case $D_k(S) = +\infty$ can be treated by choosing for each positive integer s a k-tuple of points $x_1, x_2, \ldots, x_k$ in S such that

$$\binom{k}{2}\Big\{\sum_k [d(x_i, x_j)]^{-1}\Big\}^{-1} > s$$

and then repeating the previous argument. This completes the proof of Theorem 2.1.

Theorem 2.1 shows that any set A of positive measure necessarily spreads out, not only in (ordinary) diameter, but also in harmonic diameter of any finite order.

Thus, even though A may not spread out in "volume" (measure), there is a very definite sense in which A does not remain small. This mixing character of the mixing transformations (e. g., of the polynomials $C_n (n = 0, 1, 2, \ldots)$, defined on the interval [-2, 2] by

$$C_n(x) = 2\cos(n \arccos(\frac{x}{2})), \tag{2.7}$$

which are related to the standard Čebyšev polynomials), inter alia, can be used to develop an efficient method for generating sequences of pseudo-random numbers (computer runs on selected pairs of points, using the obtained results with C_{10} as transformation ϕ, indicate that the convergence of the corresponding sequences of the second moments about the mean (variances) to the second moment of corresponding distribution function is fairly rapid).

Corollary 2.1. *Under hypotheses of Theorem 2.1, for any P - measurable subset A of S of positive measure,*

$$\lim_{k\to\infty} (\lim_{n\to\infty} D_k(\phi^n(A))) = \tau(S), \tag{2.8}$$

where D_k is the harmonic diameter of order k given by (1.6)*, and τ is the harmonic transfinite diameter given by* (1.7)*.*

Proof. The property (2.8) follows immediately from (2.1) and the fact that $\lim_{k\to\infty} D_k(S) = \tau(S)$.

3 Conclusions

There is considerable evidence (see the proof of Theorem 2.1) to support a conjecture that our result (for weakly mixing dynamical systems with discrete time) which is contained in Theorem 2.1 can be extended to weakly mixing dynamical systems with continuous time (see [3, pp. 6 - 26]).

Even though a large set cannot have a small diameter and a set with a large diameter cannot be truly small, it is still true that the diameter of a set need not be a good measure of its size and shape. A better measure is furnished by the generalised diameters (see [15, pp. 181-190 and (in Dover Edition) 295-297] and also [1], [2], [4] - [9], [11], [12], [17] and [18]).

B. Schweizer and A. Sklar have been concluded that comparison of (11.6.1) with (11.3.5) in [15] and the proof of the Rice's Theorem 1.1 quickly leads to the following conjecture [15, Problem 11.6.5]:

Problem 3.1. (*Schweizer* and *Sklar*, 1983) Does Theorem 1.1 remain valid when "diameter" is replaced by "(geometric) transfinite diameter"?

Note that our Theorem 2.1 in this paper is a first step toward the resolution of the above Problem 3.1. Also note that this problem could be formulated in a more general setting, i.e., not only for the compact set, but also for any set (with positive measure) in a general metric space.

The transfinite diameter of a compact subset A of a metric space is closely related, and often equal, to the capacity, and also to the *Čebyšev constant*. Thus we may also pose the following conjecture (open problem):

Problem 3.2. Does the Rice's Theorem 1.1 remain valid when "diameter" is replaced by "harmonic transfinite diameter"?

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Design of X bar control chart for non-normal symmetric distribution of quality characteristic

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Original scientific paper

Abstract

This paper is concerned with the design of the X bar control chart when quality characteristic has non-normal symmetric distribution. For chosen Student, standard Laplace and logistic distributions of quality characteristic, we calculated theoretical distribution of standardized sample mean and we fitted Pearson type VII distribution. Width of control limits and power of X bar control chart were established, giving evidence of goodness of fit of Pearson type VII distribution to theoretical distribution of standardized sample mean.

1 Introduction

The X bar chart is extensively used in practice to monitor a change in the process mean. It is usually assumed that quality characteristic under the surveillance of an X bar chart has a normal distribution. On the other hand, occurrence of non-normal data in industry is quite common (see [1, 8]). Violation of normality assumption results in incorrect control limits of control charts [2]. Misplaced control limits lead to inappropriate charts that will either fail to detect real changes in the process or which will generate spurious warnings when the process has not changed.

In the case of non-normal symmetric distribution of quality characteristics, no recommendations, except the use of normal distribution, are given in the quality control literature. Approximation of the distribution of sample mean with normal distribution is based on the central limit theorem, but in practice small sample sizes are usually used.

We will consider three types of non-normal symmetric distributions of quality characteristic: Student's distribution, Laplace distribution and logistic distribution. For each of these distributions, we calculated theoretical distribution of standardized sample mean (or its best approximation) and then we approximated it with normal and Pearson type VII distributions. Results based on the theoretical distribution of sample mean represent the 'golden' standard with which we will compare all other approximations.

It is presumed that a process begins in in-control state with mean μ_0 and that single assignable cause of magnitude δ results in a shift in the process mean from μ_0 to either $\mu_0 - \delta\sigma$ or $\mu_0 + \delta\sigma$, where σ is the process standard deviation [12]. It is also assumed that the standard deviation remains stable. This means that distribution of quality characteristic has shifted to the right or left for $\delta\sigma$, resulting in a change in location parameters (median, mean, mode) while scale and shape parameters remain the same. Central line of the X bar chart is set at μ_0 and upper and lower control limits, respectively, $\mu_0 + k\sigma/\sqrt{n}$ and $\mu_0 - k\sigma/\sqrt{n}$, where n represents the sample size and k width of control limits. Samples of size n are taken from the process and the sample mean is plotted on the X bar chart. If a sample mean exceeds control limits, it is assumed that some shift in the process mean has occurred and a search for the assignable cause is initiated.

The rest of the paper is organized as follows. In Sections 2, 3 and 4, respectively, description of chosen distributions of quality characteristic, distribution of standardized sample mean and Pearson type VII distribution is given. Construction of the X bar control chart and its power are examined in Section 5, along with the comparisons of theoretical distribution of sample mean with corresponding Pearson type VII distribution. Finally, conclusions are drawn in Section 6.

2 Distribution of quality characteristic

In this paper we consider three types of non-normal symmetric distributions of quality characteristic X: Student distribution, Laplace (double exponential) and logistic distribution (see [3,9,10]). All chosen distributions have heavier tails than normal distribution.

Distributions are given by their density functions. We will use following notations for mean, variance, skewness and kurtosis of quality characteristic, respectively: $\mu = E(X)$, $\sigma^2 = Var(X)$, $\alpha_3 = \frac{E(X-E(X))^3}{\sigma^{\frac{3}{2}}}$, $\alpha_4 = \frac{E(X-E(X))^4}{\sigma^4}$. As all chosen distributions are symmetric around the mean, $\alpha_3 = 0$.

1. Student's distribution $t(\nu, \mu, \eta)$

$$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\eta\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{1}{\nu}\left(\frac{x-\mu}{\eta}\right)^2\right)^{-\frac{\nu+1}{2}}, x \in \mathbb{R},$$

   where ν ($\nu \in \mathbb{R}$) is the shape parameter, η ($\eta > 0$) is scale parameter and μ ($\mu \in \mathbb{R}$) location parameter. Variance and kurtosis are, respectively, equal to

$$\sigma^2 = \frac{\nu}{\nu-2}\eta^2,\ \nu > 2, \qquad \alpha_4 = 3 + \frac{6}{\nu-4},\ \nu > 4.$$

   Random variable $Y = \frac{X-\mu}{\eta}$ has Student distribution $t(\nu)$ with density function

$$f(y) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}, y \in \mathbb{R}.$$

2. Laplace distribution $L_2(\eta, \mu)$

$$f(x) = \frac{1}{2\eta}\exp\left\{-\frac{|x-\mu|}{\eta}\right\}, \ x \in \mathbb{R},$$

where μ ($\mu \in \mathbb{R}$) is location parameter and η ($\eta > 0$) scale parameter. Variance and kurtosis are, respectively, equal to

$$\sigma^2 = 2\eta^2, \quad \alpha_4 = 6.$$

Random variable $Y = X - \mu$ has standard Laplace distribution $L_1(\eta)$ with density function

$$f_Y(y) = \frac{1}{2\eta}\exp\left\{-\frac{|y|}{\eta}\right\}, \ y \in \mathbb{R}.$$

3. Logistic distribution $LGS(\mu, \eta)$

$$f(x) = \frac{\exp\left\{-\frac{x-\mu}{\eta}\right\}}{\eta \cdot \left(1+\exp\left\{-\frac{x-\mu}{\eta}\right\}\right)^2}, \ x \in \mathbb{R},$$

where μ ($\mu \in \mathbb{R}$) is the location parameter and η ($\eta > 0$) scale parameter. Variance and kurtosis are, respectively, equal to

$$\sigma^2 = \frac{\eta^2\pi^2}{3}, \quad \alpha_4 = 4.2.$$

Random variable $Y = \frac{X-\mu}{\eta}$ has standard logistic distribution with density function

$$f_Y(y) = \frac{\mathrm{e}^{-y}}{(1+\mathrm{e}^{-y})^2}, \ y \in \mathbb{R},$$

3 Distribution of standardized sample mean

3.1 Sample from Student's distribution

Witkowský [13,14] proposed a method for numerical evaluation of the distribution function and the density function of a linear combination of independent Student's t random variables. The method is based on the inversion formula which leads to the one-dimensional numerical integration.

Let $(X_1, X_2, \ldots, X_n)$ be a sample from Student's $t(\nu, \mu, \lambda)$ distribution. Then $Y_k = \frac{X_k-\mu}{\eta}$, $k = 1, 2, \ldots, n$ are independent Student's t_ν random variables with ν degrees of freedom. Further, let $Y = \sum_{k=1}^n Y_k$ be sum of these variables and let

$\phi_{Y_k}(t)$ denote the characteristic function of Y_k. The characteristic function of Y is

$$\phi_Y(t) = \prod_{k=1}^{n} \phi_{Y_k}(t),$$

where

$$\phi_{Y_k}(t) = \frac{1}{2^{\frac{\nu}{2}-1}\Gamma(\frac{\nu}{2})} \left(\nu^{\frac{1}{2}}|t|\right)^{\frac{\nu}{2}} K_{\nu/2}\left(\nu^{\frac{1}{2}}|t|\right),$$

where $K_\alpha(z)$ denotes modified Bessel function of the second kind.

The cumulative distribution function (cdf) $F_Y(y)$ of random variable Y is according to the inversion formula due to Gil-Pelaez [5] given by

$$\begin{aligned} F_Y(y) &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{\mathrm{e}^{-ity}\phi_Y(t)}{t}\right)\mathrm{d}t = \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\sin(ty)\phi_Y(t)}{t}\,\mathrm{d}t \qquad (1) \end{aligned}$$

and the probability density function (pdf) $f_Y(y)$ of Y is given by

$$\begin{aligned} f(y) &= \frac{1}{\pi}\int_0^\infty \mathrm{Re}(\mathrm{e}^{-ity}\phi_Y(t))\,\mathrm{d}t = \\ &= \frac{1}{\pi}\int_0^\infty \cos(ty)\phi_Y(t)\,\mathrm{d}t \qquad (2) \end{aligned}$$

For any chosen y algorithm *tdist* in *R* package *tdist* [15] evaluates the integrals in (1) and (2) by multiple p-points Gaussian quadrature over the real interval $t \in (0, 10\pi)$. The whole interval is divided into m subintervals given by prespecified limits and the integration over each subinterval is done with p-points Gaussian quadrature which involves base points b_{ij}, and weight factors w_{ij}, $i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, m$. So,

$$\begin{aligned} F_Y(y) &\approx \frac{1}{2} + \frac{1}{\pi}\sum_{j=1}^{m}\sum_{i=1}^{p}\frac{\sin(b_{ij}y)}{b_{ij}} w_{ij}\phi_Y(b_{ij}) = \\ &= \frac{1}{2} + \frac{1}{\pi}\sum_{j=1}^{m}\sum_{i=1}^{p}\frac{\sin(b_{ij}x)}{b_{ij}} W_{ij}, \\ f_Y(y) &\approx \frac{1}{\pi}\sum_{j=1}^{m}\sum_{i=1}^{p}\cos(b_{ij}) w_{ij}\phi_Y(b_{ij}) = \\ &= \frac{1}{\pi}\sum_{j=1}^{m}\sum_{i=1}^{p}\cos(b_{ij}) W_{ij}, \end{aligned}$$

where $W_{ij} = w_{ij}\phi_Y(b_{ij})$. For evaluation of $F_Y(y)$ and $f_Y(y)$ in many different points the algorithm requires only one evaluation of the weights W_{ij}, for

$i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, m$, which directly depends on the characteristic function $\phi_Y(\cdot)$ and does not depend on y.

Then, distribution function and density function of standardized sample mean $T_n = \frac{\overline{X}-\mu}{\sigma}\sqrt{n} = Y\sqrt{\frac{\nu-2}{n\nu}}$ are equal to

$$\begin{aligned} F_{T_n}(t) &= P\left\{Y\sqrt{\frac{\nu-2}{n\nu}} \leq t\right\} = F_Y\left(\sqrt{\frac{n\nu}{\nu-2}}t\right) \\ f_{T_n}(t) &= \sqrt{\frac{n\nu}{\nu-2}} \cdot f_Y\left(\sqrt{\frac{n\nu}{\nu-2}}t\right), \ t \in \mathbb{R} \end{aligned}$$

Skewness and kurtosis of standardized sample mean are, respectively, equal

$$\alpha_{3,T_n} = 0, \quad \alpha_{4,T_n} = 3 + \frac{6}{n(\nu-4)}.$$

3.2 Sample from Laplace distribution

Let $(X_1, X_2, \ldots, X_n)$ be a sample from Laplace distribution $L_2(\mu, \eta)$. Then, $Y_k = X_k - \mu, \ k = 1, 2, \ldots, n$ are independent random variables with standard Laplace distribution $L_1(\eta)$.

Statement 3.1. *If U and V are two independent random variables with exponential $\varepsilon(\eta)$ distribution, then $Y = U - V$ has standard Laplace distribution.*

Standard exponential distribution is gamma distribution, $\Gamma(1, \eta)$. Sum of n independent variables with $\Gamma(1, \eta)$ distribution is gamma distribution $\Gamma(n, \eta)$. In that way, using Statement 1, we conclude that sum of n independent random variables $Y_1, Y_2, \ldots, Y_n$ with standard Laplace distribution can be written as the difference of two random variables with gamma distribution $\Gamma(n, \eta)$ which is called *bilateral gamma distribution.*

Bilateral gamma distribution is symmetric around 0, with density function for $y > 0$ [11]

$$f(y) = \left(\frac{\eta}{2}\right)^n \cdot \frac{1}{(n-1)!}\sum_{k=0}^{n-1} a_k y^k \exp\{-\eta y\},$$

where the coefficients $(a_k)_{k=0,\ldots,n-1}$ are given by

$$a_k = \binom{n-1}{k}\frac{1}{(2\eta)^{n-1-k}}\prod_{l=0}^{n-2-k}(n+l), \ a_{n-1} = 1.$$

Probability distribution function, for $y > 0$ is

$$F_Y(y) = \frac{1}{2} + \left(\frac{\eta}{2}\right)^n \cdot \frac{1}{(n-1)!}\sum_{k=0}^{n}\frac{a_k}{\eta^{k+1}}\gamma(k+1, \eta x),$$

where $\gamma(n,x)$ is incomplete gamma function.

Then, distribution function and density function of standardized sample mean $T_n = \frac{\overline{X}-\mu}{\sigma}\sqrt{n} = \frac{Y}{\eta\cdot\sqrt{2n}}$ are equal to

$$
\begin{aligned}
F_{T_n}(t) &= P\left\{\frac{Y}{\eta\cdot\sqrt{2n}} \le t\right\} = F_Y\left(\eta\sqrt{2n}t\right),\\
f_{T_n}(t) &= \eta\sqrt{2n}\cdot f_Y\left(\eta\sqrt{2n}t\right),\ t\in\mathbb{R}
\end{aligned}
$$

Skewness and kurtosis of standardized sample mean are, respectively, equal

$$\alpha_{3,T_n} = 0, \quad \alpha_{4,T_n} = 3 + \frac{3}{n}.$$

3.3 Sample from logistic distribution

Let $(X_1, X_2, \ldots, X_n)$ be a random sample from logistic distribution $LGS(\mu,\eta)$.

George and Mudholkar [6] showed that a standardized Student's t-distribution provides a very good approximation for the distribution of a convolution of n i.i.d. logistic variables. These authors then compared three approximations: (1) standard normal approximation, (2) Edgeworth series approximation correct to order n^{-1}, and (3) Student's t approximation with $\nu = 5n+4$ degrees of freedom. They showed that the third provides a very good approximation.

Gupta and Han [7] considered the Edgeworth series expansions up to order n^{-3} for the distribution of the standardized sample mean

$$T_n = \frac{\overline{X}-\mu}{\sigma}\sqrt{n}.$$

Distribution function is given by

$$
\begin{aligned}
F_{T_n}(t) &\approx \Phi(t) - \varphi(t)\left(\frac{1}{n}\left(\frac{1}{4!}\cdot\frac{6}{5}H_3(t)\right) + \frac{1}{n^2}\left(\frac{1}{6!}\cdot\frac{48}{7}H_5(t) + \right.\right.\\
&+ \left.\left.\frac{35}{8!}\left(\frac{6}{5}\right)^2 H_7(t)\right)\right) + \frac{1}{n^3}\left(\frac{1}{8!}\cdot\frac{432}{5}H_7(t) + \frac{210}{10!}\frac{48}{7}\cdot\frac{6}{5}H_9(t) + \right.\\
&+ \left.\frac{5775}{12!}\left(\frac{6}{5}\right)^3 H_{11}(t)\right)\Bigg),\ t\in\mathbb{R},
\end{aligned}
$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are standard normal pdf and cdf and $H_j(x)$ is the Hermite polynomial.

Then, probability density function of standardized sample mean is

$$f_{T_n}(t) \approx \varphi(t)\left(1 + \frac{1}{n}\cdot\frac{1}{4!}\cdot\frac{6}{5}\cdot\left(tH_3(t) - H_3'(t)\right) + \right.$$

$$
\begin{aligned}
&+ \quad \frac{1}{n^2}\left(\frac{1}{6!}\cdot\frac{48}{7}\cdot\left(tH_5(t)-H_5^{'}(t)\right)+\frac{35}{8!}\left(\frac{6}{5}\right)^2\left(tH_7(t)-H_7^{'}(t)\right)\right)+\\
&+ \quad \frac{1}{n^3}\left(\frac{1}{8!}\cdot\frac{432}{5}\cdot\left(tH_7(t)-H_7^{'}(t)\right)+\frac{210}{10!}\frac{48}{7}\cdot\frac{6}{5}\left(tH_9(t)-H_9^{'}(t)\right)+\right.\\
&+ \quad \left.\frac{5775}{12!}\left(\frac{6}{5}\right)^3\left(tH_{11}(t)-H_{11}^{'}(t)\right)\right)\bigg),\ t\in\mathbb{R}
\end{aligned}
$$

Gupta and Han [7] compared this approximation with the approximations mentioned earlier, and they showed the approximation to be far better than even the Student's t-approximation suggested by George and Mudholkar [6].

Skewness and kurtosis of standardized sample mean are, respectively, equal

$$\alpha_{3,T_n}=0,\quad \alpha_{4,T_n}=3+\frac{1.2}{n}.$$

4 Pearson type VII distribution

Probability density function is equal to (see [3, 9])

$$f(x)=\frac{1}{\alpha B\left(m-\frac{1}{2},\frac{1}{2}\right)}\cdot\left(1+\frac{x^2}{\alpha^2}\right)^{-m},\ x\in\mathbb{R},$$

where

$$m = \frac{5\alpha_4-9}{2(\alpha_4-3)}, \tag{3}$$

$$\alpha = \sqrt{\frac{2\alpha_4}{\alpha_4-3}} \tag{4}$$

and $B(a,b)$ is beta function.

Then, cumulative distribution function is equal to

$$F(x)=\frac{1}{2}I_{\alpha^2/(\alpha^2+x^2)}\left(m-\frac{1}{2},\frac{1}{2}\right),$$

for $x<0$ and

$$F(x)=1-\frac{1}{2}I_{\alpha^2/(\alpha^2+x^2)}\left(m-\frac{1}{2},\frac{1}{2}\right),$$

for $x>0$, where $I_x(a,b)=\frac{B_x(a,b)}{B(a,b)}$ and $B_x(a,b)$ is incomplete beta function.

5 Design of X bar control chart

We considered following distributions of quality characteristic: Student's $t(10)$ distribution with 10 degrees of freedom, standard Laplace $L(1)$ and logistic $LGS(1)$

distribution. We calculated theoretical distribution of standardized sample mean of considered distributions for sample sizes $n = 3 - 10$, using results from Section 3 and then we approximated it with normal and Pearson type VII distributions. Parameters of the fitted Pearson type VII distribution are calculated using formulas (3) and (4). Code for all calculations was written, by the author, in statistical software R. Width of control limits of the X bar control chart is calculated from

$$\alpha = 1 - P\{\mu_0 - k\frac{\sigma}{\sqrt{n}} \leq \bar{X} \geq \mu_0 + k\frac{\sigma}{\sqrt{n}} | \mu = \mu_0\} = 2(1 - F(k)), \quad (5)$$

for the probability of false alarms $\alpha = 0.0027$ and cumulative distribution function F of standardized sample mean, using Brent's root-finding method [4].

Calculated widths of control limits, for considered distributions of quality characteristic, sample sizes $n = 3 - 10$, probability of false alarms $\alpha = 0.0027$, for theoretical distribution of standardized sample mean and Pearson type VII distribution, are given in Table 1.

Sample size	**Width of control limits**					
	Student $t(10)$		*Laplace* $L(1)$		*Logistic* $LGS(1)$	
	Theor.	Pearson	Theor.	Pearson	Theor.	Pearson
$n = 3$	3.21966	3.22227	3.54221	3.53915	3.25580	3.26074
$n = 4$	3.16998	3.17156	3.43224	3.43628	3.20035	3.20234
$n = 5$	3.13867	3.13966	3.36034	3.36606	3.16405	3.16527
$n = 6$	3.11712	3.11775	3.30939	3.31520	3.13877	3.13966
$n = 7$	3.10136	3.10178	3.27130	3.27668	3.12021	3.12091
$n = 8$	3.08934	3.08962	3.24168	3.24652	3.10602	3.10660
$n = 9$	3.07987	3.08005	3.21796	3.22227	3.09482	3.09531
$n = 10$	3.07221	3.07233	3.19852	3.20234	3.08577	3.08619

Table 1: Width of control limits of X bar control chart

As it can be seen in the Table 1, values of the width of the control limits calculated from theoretical distribution and corresponding Pearson type VII distribution are very close, i.e. Pearson type VII distribution fit very well to theoretical distribution of standardized sample mean. On the other hand, normal approximation would give value of $k = 2.99998$, for all n and all distributions of quality characteristics.

Now, we are interested to see what is the power of X bar control charts for detecting shifts from $\delta = 0.5 - 3(0.5)$, for calculated width of control limits. Power of X bar control chart for detecting shifts from mean μ_0 to $\mu_1 = \mu_0 \pm \delta\sigma$ can be calculated from

$$1-\beta = 1-P\{\mu_0-k\frac{\sigma}{\sqrt{n}} \leq \bar{X} \geq \mu_0+k\frac{\sigma}{\sqrt{n}} | \mu = \mu_1\} = F(-k-\delta\sqrt{n})+F(-k+\delta\sqrt{n}).$$

Mainly, we want to investigate what is the minimum positive shift that X bar chart can detect with a power of at least 90%.

Calculated power of X bar control chart, for considered distributions of quality characteristic, sample sizes $n = 3 - 10$, shifts $\delta = 0.5 - 3(0.5)$ for both theoretical distribution of standardized sample mean and corresponding Pearson type VII distribution, are given in Table 2.

From the Table 2, we see that X bar control chart can detect shifts of $\delta = 1.5$ with power of at least 90% for sample sizes of at least $n = 9$ for Student distribution with 10 degrees of freedom, standard Laplace and logistic distributions. In order for the X bar chart to detect shifts of $\delta = 2.0$ with power of 90% and greater, it is necessary to take samples of size at least $n = 4$ for all chosen distributions of quality characteristics. Also, we can once more notice that Pearson type VII distribution approximate distribution of standardized sample mean rather well. In general, it can be concluded that X bar chart can detect shifts of at least $\delta = 1.5$ with power of 90% and greater, for non-normal symmetric distribution of quality characteristic.

6 Conclusions

We considered design of X bar control chart when quality characteristic has one of the following non-normal symmetric distributions: Student's distribution with 10 degrees of freedom, logistic distribution and standard Laplace distribution. We calculated theoretical distribution of standardized sample mean (or its best approximation) and approximated it with Pearson type VII distribution. Then we calculated width of control limits of X bar chart, which gave evidence of goodness of fit of Pearson type VII distribution to theoretical distribution of standardized sample mean. Further, we examined the power of X bar control chart in detecting the shifts. Results suggest that X bar chart can detect shifts of at least $\delta = 1.5$ with power of 90% and greater.

Although our results of design of X-bar chart are based only on three types of non-normal symmetric distributions, we would recommend use of Pearson type VII distribution in general case, when distribution of quality characteristic is non-normal but symmetric. Other cases will be in scope of our future research.

Distribution	Power											
	Theor.	Pearson	Theor.	Pearson	Theor.	Pearson	Theor.	Pearson	Theor.	Pearson	Theor.	Pearson
	$\delta = 0.5$		$\delta = 1.0$		$\delta = 1.5$		$\delta = 2.0$		$\delta = 2.5$		$\delta = 3.0$	
$t(10)$												
$n = 3$	0.01099	0.01099	0.06651	0.06618	0.26087	0.25974	0.59983	0.59894	0.87154	0.87114	0.97504	0.97482
$n = 4$	0.01620	0.01621	0.11751	0.11712	0.43071	0.42999	0.80164	0.80138	0.96611	0.96594	0.99674	0.99671
$n = 5$	0.02243	0.02243	0.17950	0.17911	0.58712	0.58681	0.91083	0.91070	0.99188	0.99182	0.99959	0.99960
$n = 6$	0.02962	0.02962	0.24878	0.24845	0.71448	0.71438	0.96254	0.96245	0.99816	0.99815	0.99995	0.99995
$n = 7$	0.03775	0.03774	0.32185	0.32161	0.81000	0.80998	0.98503	0.98498	0.99960	0.99960	0.99999	0.99999
$n = 8$	0.04677	0.04676	0.39566	0.39551	0.87751	0.87750	0.99424	0.99422	0.99991	0.99992	1	1
$n = 9$	0.05665	0.05664	0.46776	0.46767	0.92312	0.92309	0.99785	0.99785	0.99998	0.99998	1	1
$n = 10$	0.06736	0.06735	0.53630	0.53626	0.95283	0.95280	0.99922	0.99922	1	1	1	1
$L(1)$												
$n = 3$	0.00767	0.00710	0.03700	0.03554	0.15415	0.15877	0.46420	0.46740	0.80609	0.80137	0.95142	0.95318
$n = 4$	0.01114	0.01042	0.07178	0.07084	0.31786	0.32065	0.73166	0.72579	0.94334	0.94380	0.99163	0.99210
$n = 5$	0.01555	0.01475	0.12146	0.12123	0.49730	0.49494	0.87609	0.87397	0.98414	0.98463	0.99861	0.99858
$n = 6$	0.02093	0.02011	0.18418	0.18424	0.65106	0.64692	0.94518	0.94510	0.99571	0.99578	0.99978	0.99973
$n = 7$	0.02729	0.02650	0.25608	0.25593	0.76703	0.76377	0.97648	0.97671	0.99887	0.99883	0.99997	0.99995
$n = 8$	0.03465	0.03391	0.33272	0.33208	0.84872	0.84688	0.99015	0.99027	0.99971	0.99967	0.99999	0.99999
$n = 9$	0.04300	0.04231	0.41006	0.40891	0.90386	0.90305	0.99595	0.99597	0.99993	0.99991	1	1
$n = 10$	0.05231	0.05169	0.48497	0.48348	0.93996	0.93972	0.99837	0.99834	0.99998	0.99997	1	1
$LGS(1)$												
$n = 3$	0.01065	0.01033	0.06193	0.06130	0.24689	0.24604	0.58637	0.58397	0.86530	0.86384	0.97262	0.97258
$n = 4$	0.01552	0.01531	0.11088	0.11064	0.41778	0.41720	0.79448	0.79351	0.96368	0.96368	0.99624	0.99627
$n = 5$	0.02148	0.02133	0.17185	0.17176	0.57753	0.57690	0.90720	0.90697	0.99107	0.99111	0.99953	0.99952
$n = 6$	0.02845	0.02832	0.24091	0.24084	0.70791	0.70739	0.96070	0.96070	0.99794	0.99794	0.99994	0.99994
$n = 7$	0.03638	0.03627	0.31430	0.31420	0.80561	0.80528	0.98413	0.98416	0.99955	0.99954	0.99999	0.99999
$n = 8$	0.04523	0.04514	0.38874	0.38860	0.87457	0.87441	0.99383	0.99384	0.99990	0.99990	1	1
$n = 9$	0.05496	0.05488	0.46161	0.46144	0.92114	0.92108	0.99767	0.99767	0.99998	0.99998	1	1
$n = 10$	0.06554	0.06548	0.53094	0.53076	0.95150	0.95149	0.99915	0.99914	1	1	1	1

Table 2: Power of X bar control chart

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Optimizacija površine u konveksnim cjelobrojnim petouglovima

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Apstrakt

Površina konveksnog petougla je funkcija od zbira s površina pet njegovih ivičnih trouglova i cikličnog zbira q proizvoda površina pet parova susjednih ivičnih trouglova. Ovo je tvrđenje Gausove formule za petougao $P^2 - sP + q = 0$ iz 1823. godine, koja se u literaturi pominje rijetko. U ovom radu se razmatraju problemi optimizacije među veličinama s, q i P na skupu konveksnih cjelobrojnih petouglova. Pri dokazima tvrđenja često se koristi gore pomenuta Gausova formula.

1 Uvod

Definicija 1. Neka je dat pravougli koordinatni sistem u ravni. Tačku čije su obje koordinate cjelobrojne zovemo **cjelobrojna tačka**. Poligon čija su tjemena cjelobrojne tačke zovemo **cjelobrojni poligon**. Cjelobrojni poligon čiji su svi unutrašnji uglovi manji od 180° zovemo **konveksan cjelobrojni poligon**.

Kako za površinu trougla sa tjemenima u tačkama $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ važi formula

$$P_{ABC} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|,$$

odavde slijedi da površina cjelobrojnog trougla nije manja od 1/2. Za cjelobrojne trouglove površine 1/2, tj. minimalne površine, uvodimo poseban naziv.

Definicija 2. Cjelobrojan trougao površine 1/2 naziva se **fundamentalan** trougao.

Za proučavanje površine konveksnih cjelobrojnih poligona posebno važnu ulogu imaju preslikavanja ravni koja fundamentalan trougao preslikavaju u fundamentalan trougao. Ovakva preslikavanja čuvaju površinu konveksnog cjelobrojnog poligona i potpuno su određena jednom cjelobrojnom kvadratnom matricom drugog reda čija determinanta pripada skupu $\{-1, 1\}$.

Definicija 3. Za kvadratnu matricu V kažemo da je **unimodularna** ako je $\det V \in \{-1, 1\}$. Za linearno preslikavanje kažemo da je **unimodularno** ako je njegova matrica (u standardnoj bazi prostora $\mathbb{R}^2$) unimodularna. Za linearno preslikavanje kažemo da je **cjelobrojno unimodularno** ako je njegova matrica cjelobrojna i unimodularna.

Definicija 4. Kompoziciju unimodularnog preslikavanja i translacije zovemo **unimodularno afino preslikavanje**. Kompoziciju cjelobrojnog unimodularnog preslikavanja i cjelobrojne translacije (translacije za cjelobrojan vektor) zovemo **cjelobrojno unimodularno afino preslikavanje** ili **cjelobrojna ekvivalencija**. Za dva cjelobrojna poligona kažemo da su **cjelobrojno ekvivalentni** ako postoji cjelobrojna ekvivalencija koja preslikava jedan poligon u drugi.

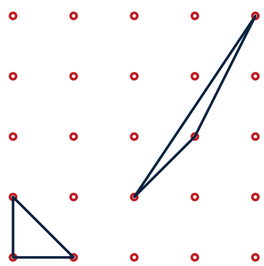
Najvažnije osobine cjelobrojnog unimodularnog preslikavanja date su u sljedećoj teoremi ([7]).

Teorema 1. *a) Cjelobrojno unimodularno preslikavanje čuva broj cjelobrojnih tačaka kompaktnog konveksnog skupa.*

b) Kompozicija dva cjelobrojna unimodularna preslikavanja je cjelobrojno unimodularno preslikavanje.

Iz ove teoreme neposredno slijedi sljedeća posljedica.

Posljedica 1. *Svaka dva fundamentalna trougla su cjelobrojno ekvivalentna. Specijalno je svaki fundamentalni trougao cjelobrojno ekvivalentan sa trouglom čija su tjemena* $(0, 0)$*,* $(1, 0)$ *i* $(0, 1)$*.*



Slika 1:

2 Površina konveksnog petougla

Više autora (Rabinowitz u [7], Simpson u [8], Matić-Kekić u [5], Govedarica u [4], Bárány i Tokushige u [1]) se bavilo problemom određivanja minimalne površine konveksnog cjelobrojnog n–ugla. U tom smislu za konveksan cjelobrojan petougao važi sljedeće tvrđenje.

Teorema 2. *Minimalna površina konveksnog cjelobrojnog petougla je* $5/2$. *Svaki konveksan cjelobrojan petougao minimalne površine je cjelobrojno ekvivalentan sa petouglom na slici 1.*

Neka je dat konveksan petougao $X = A_1A_2A_3A_4A_5$.

Definicija 5. Trouglove $A_{i-1}A_iA_{i+1}$, $i = 1,2,3,4,5$ ($A_0 \equiv A_5$, $A_6 \equiv A_1$), zvaćemo **ivičnim trouglovima** petougla X. Dva ivična trougla su **susjedna** ako imaju zajedničku stranicu. Cjelobrojan trougao je **fundamentalan** ako mu je površina jednaka $1/2$.

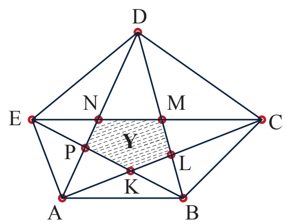
Uvedimo sljedeće oznake:

- s_i – površina trougla $A_{i-1}A_iA_{i+1}$ $\quad (i \in \{1,2,3,4,5\},\ A_0 \equiv A_5,\ A_6 \equiv A_1)$,
- $s = s_1 + s_2 + s_3 + s_4 + s_5$,
- $q = s_1s_2 + s_2s_3 + s_3s_4 + s_4s_5 + s_5s_1$,
- P površina petougla X.

Tada važi sljedećih pet tvrđenja.

Stav 1. $2P > s$.

Dokaz. Označimo sa Y i Z redom konveksan petougao $ABCDE$ i petokraku zvijezdu $AKBLCMDNEP$ koje obrazuju dijagonale datog konveksnog petougla X (slika 2).



Slika 2:

Kako je $s + P_Y + P_Z = 2P$, slijedi da je $s < 2P$. $\square$

Stav 2. $P^2 - sP + q = 0$.

Dokaz. Koristićemo identitet

$$\sin\varphi_1 \sin\varphi_3 + \sin\varphi_2 \sin(\varphi_1 + \varphi_2 + \varphi_3) = \sin(\varphi_1 + \varphi_2) \cdot \sin(\varphi_2 + \varphi_3). \qquad (1)$$

(koji slijedi iz sljedećih jednakosti

$$\cos(\varphi_1 - \varphi_3) - \cos(\varphi_1 + 2\varphi_2 + \varphi_3) = 2\sin(\varphi_1 + \varphi_2) \cdot \sin(\varphi_2 + \varphi_3),$$

$$(\cos(\varphi_1 - \varphi_3) - \cos(\varphi_1 + \varphi_3)) + (\cos(\varphi_1 + \varphi_3) - \cos(\varphi_1 + 2\varphi_2 + \varphi_3)) =$$
$$= 2\sin(\varphi_1 + \varphi_2) \cdot \sin(\varphi_2 + \varphi_3),$$

$$\sin\varphi_1 \sin\varphi_3 + \sin\varphi_2 \sin(\varphi_1 + \varphi_2 + \varphi_3) = \sin(\varphi_1 + \varphi_2) \cdot \sin(\varphi_2 + \varphi_3).)$$

Neka je $A_iA_5 = b_i,\ i = 1,2,3,4,$ i $\sphericalangle A_iA_5A_{i+1} = \varphi_i,\ i = 1,2,3$. Kako je

$$\frac{1}{2}\,b_1b_2\sin\varphi_1 = s_1, \quad \frac{1}{2}\,b_3b_4\sin\varphi_3 = s_4, \quad \frac{1}{2}\,b_2b_3\sin\varphi_2 = P_{A_2A_3A_5} = P - s_1 - s_4,$$
$$\frac{1}{2}\,b_1b_4\sin(\varphi_1 + \varphi_2 + \varphi_3) = s_5, \quad \frac{1}{2}\,b_1b_3\sin(\varphi_1 + \varphi_2) = P_{A_5A_1A_3} = P - s_2 - s_4,$$
$$\frac{1}{2}\,b_2b_4\sin(\varphi_2 + \varphi_3) = P_{A_4A_5A_2} = P - s_1 - s_3,$$

množenjem jednakosti (1) sa $\frac{1}{4}\,b_1b_2b_3b_4$ dobijamo da je

$$s_1s_4 + (P - s_1 - s_4)s_5 = (P - s_2 - s_4)(P - s_1 - s_3),$$

što je ekvivalentno sa $P^2 - sP + q = 0$.

Posljedica 2. $2P = s + \sqrt{s^2 - 4q}$.

Dokaz. Slijedi iz prethodna dva stava, jer zbog $2P > s$ ne može biti $2P = s - \sqrt{s^2 - 4q}$. □

Stav 3. *U svakom konveksnom petouglu $A_1A_2A_3A_4A_5$ postoji i takvo da je $P < s - s_i$.*

Dokaz. Neka je $s_1 = \min s_i$ i $A_2A_4 \cap A_3A_5 = B$ ([6]). Tada je

$$P < P_{A_5A_1A_2B} + P_{A_2A_3A_4} + P_{A_3A_4A_5} = P_{A_5A_1A_2B} + s_3 + s_4,$$

a kako je

$$P_{A_5A_1A_2B} = P_{A_5A_1B} + P_{A_1A_2B} \leq P_{A_4A_5A_1} + P_{A_1A_2A_3} = s_5 + s_2,$$

slijedi da je $P < s_5 + s_2 + s_3 + s_4$, tj. $P < s - s_1$. □

Dalje ćemo smatrati da je petougao X još i cjelobrojan, tj. razmatraćemo skup konveksnih cjelobrojnih petouglova.

Posljedica 3. *U svakom konveksnom cjelobrojnom petouglu broj $4(s^2 - 4q)$ je potpun kvadrat.*

Dokaz. Na osnovu prethodne propozicije je $4P - 2s = \sqrt{4(s^2 - 4q)}$, pa kako su $4P$ i $2s$ cijeli brojevi slijedi da je $4(s^2 - 4q)$ potpun kvadrat. □

Iz ove propozicije slijedi da kod konveksnog cjelobrojnog petougla svi ivični trouglovi ne mogu biti fundamentalni. Zaista, u tom slučaju bi bilo $s = 5/2$ i $q = 5/4$, tj. $4(s^2 - 4q) = 5$, što očito nije potpun kvadrat.

3 Optimizacija veličina q i P u odnosu na s

Naš cilj je da odredimo minimalnu vrijednost od s na skupu konveksnih cjelobrojnih petouglova, da odredimo skup svih konveksnih cjelobrojnih petouglova sa ovom minimalnom vrijednošću od s i da uporedimo ovaj skup sa skupom konveksnih cjelobrojnih petouglova minimalne površine.

Stav 4. *Ako su u konveksnom cjelobrojnom petouglu tri ivična trougla fundamentalna, ti trouglovi su susjedni.*

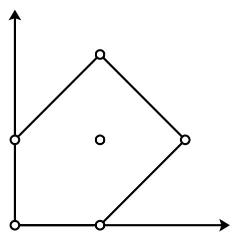
Dokaz. Pretpostavimo da postoji konveksan cjelobrojan petougao kod koga je

$$s_2 = s_4 = s_5 = \frac{1}{2},\ s_1 = \frac{k}{2},\ s_3 = \frac{l}{2} \quad (k, l \in \mathbb{N}).$$

Tada je $s = \dfrac{k+l+3}{2}$, $q = \dfrac{2(k+l)+1}{4}$ pa je $4(s^2 - 4q) = (k+l-1)^2 + 4$. Međutim, pošto je $k+l-1$ prirodan broj to $(k+l-1)^2+4$ ne može biti potpun kvadrat. Kontradikcija. □

Posljedica 4. *U svakom konveksnom cjelobrojnom petouglu postoji i takvo da je $s_i \geq 1$ i $s_{i+1} \geq 1$, tj. postoje dva susjedna ivična trougla koja nisu fundamentalna.*

Posljedica 5. *U svakom konveksnom cjelobrojnom petouglu je $s \geq 7/2$.*



Slika 3:

Stav 5. *Postoji, do na cjelobrojnu ekvivalentnost jedinstven, konveksan cjelobrojan petougao kod koga je*

$$s_1 = s_2 = 1,\ s_3 = s_4 = s_5 = 1/2.$$

Dokaz. U svakom takvom konveksnom cjelobrojnom petouglu X je $s = 7/2$ i $q = 5/2$ pa je na osnovu posljedice 2 njegova površina $P = 5/2$. Na osnovu teoreme 2 slijedi da je X konveksan cjelobrojan petougao minimalne površine i da je cjelobrojno ekvivalentan sa petouglom na slici 3. □

Ovim je dokazana sljedeća teorema.

Teorema 3. (a) *Minimalna vrijednost od s na skupu konveksnih cjelobrojnih petouglova je $7/2$.*

(b) *Svaki konveksan cjelobrojan petougao kod koga je* $s = 7/2$ *je cjelobrojno ekvivalentan sa petouglom na slici 1.*

(c) *Skup konveksnih cjelobrojnih petouglova sa minimalnom vrijednošću od* s *se podudara sa skupom konveksnih cjelobrojnih petouglova minimalne površine.*

Da bismo odredili, samo u funkciji od s, (najbolju) donju i gornju granicu za površinu P konveksnog cjelobrojnog petougla, na osnovu posljedice 2 ćemo u funkciji od s naći (najbolju) donju i gornju granicu za q.

Stav 6. *U konveksnom cjelobrojnom petouglu* X *važe sljedeća tvrđenja.*

(a) $s - 1 \leq q \leq \dfrac{4s^2 - 4s + 5}{16}$.

(b) *Jednakost* $q = s - 1$ *važi ako i samo ako postoji prirodan broj* k *takav da je petougao* X *cjelobrojno ekvivalentan sa petouglom čija su tjemena* $A_1(1, 0)$, $A_2(k + 1, 2)$, $A_3(k, 2)$, $A_4(0, 1)$, $A_5(0, 0)$.

(c) *Jednakost* $q = \dfrac{4s^2 - 4s + 5}{16}$ *važi ako i samo ako postoje prirodni brojevi* m *i* l *takvi da je petougao* X *cjelobrojno ekvivalentan sa petouglom čija su tjemena* $A_1(m + 1, 1)$, $A_2(l, m + 1)$, $A_3(0, 1)$, $A_4(0, 0)$, $A_5(1, 0)$.

(d) *Jednakosti* $s - 1 = q = \dfrac{4s^2 - 4s + 5}{16}$ *važe ako i samo ako je petougao* X *cjelobrojno ekvivalentan sa petouglom na slici 1.*

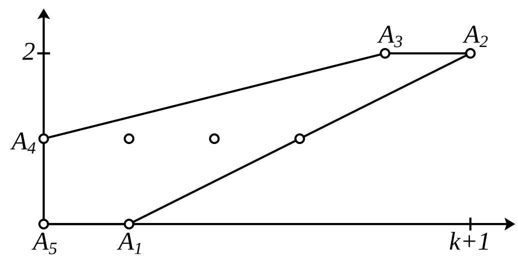
Dokaz. Na osnovu posljedice 4 postoji i takvo da je $\left(s_i - \frac{1}{2}\right)\left(s_{i+1} - \frac{1}{2}\right) \geq \frac{1}{4}$ pa je

$$\left(s_1 - \frac{1}{2}\right)\left(s_2 - \frac{1}{2}\right) + \left(s_2 - \frac{1}{2}\right)\left(s_3 - \frac{1}{2}\right) + \left(s_3 - \frac{1}{2}\right)\left(s_4 - \frac{1}{2}\right) +$$
$$+ \left(s_4 - \frac{1}{2}\right)\left(s_5 - \frac{1}{2}\right) + \left(s_5 - \frac{1}{2}\right)\left(s_1 - \frac{1}{2}\right) \geq \frac{1}{4},$$

što je ekvivalentno sa $q \geq s - 1$.

Neka je $q = s - 1$. Tada iz prethodnog slijedi da možemo uzeti da je $s_1 = s_2 = 1$, $s_3 = s_5 = \frac{1}{2}$ i $s_4 = \frac{k}{2}$, $k \in \mathbb{N}$, tj. da je $(s_1, s_2, s_3, s_4, s_5) = \left(1, 1, \frac{1}{2}, \frac{k}{2}, \frac{1}{2}\right)$. Za svaki prirodan broj k postoji konveksan cjelobrojan petougao sa ovom petorkom $(s_1, s_2, s_3, s_4, s_5)$.
Zaista, kako je trougao $A_4A_5A_1$ fundamentalan, na osnovu posljedice 1 možemo uzeti da je $A_4(0, 1)$, $A_5(0, 0)$, $A_1(1, 0)$ (što znači da smo pretpostavili da je petougao pozitivno orijentisan). Zbog $s_1 = 1$ i $s_4 = \frac{k}{2}$ slijedi da postoje prirodni

brojevi x, y takvi da je $A_2(x,2)$ i $A_3(k,y)$ (slika 2). Dalje iz $s_2 = 1$ i $s_3 = \frac{1}{2}$ dobijamo jednačine $xy - y = 2k$ i $xy - x = k+1$. Eliminišući k iz ove dvije jednačine dobijamo da je $y(x+1) = 2(x+1)$ pa je $y = 2$ i dalje $x = k+1$, $A_2(k+1,2)$ i $A_3(k,2)$.
Dakle, svaki konveksan cjelobrojan petougao kod koga je $q = s - 1$ je za neki prirodan broj k cjelobrojno ekvivalentan sa petouglom čija su tjemena $A_1(1,0)$, $A_2(k+1,2)$, $A_3(k,2)$, $A_4(0,1)$, $A_5(0,0)$ (slika 4).



Slika 4:

Navedimo još jedan dokaz nejednakosti $s - 1 \leq q$.

Na osnovu stava 3 postoji i takvo da je $P < s - s_i$, odakle dobijamo redom

$$s_i + P + \frac{1}{2} \leq s,$$

$$2s_i + s + \sqrt{s^2 - 4q} + 1 \leq 2s,$$

$$2s_i \leq s - 1 - \sqrt{s^2 - 4q}.$$

Kako je $2s_i \geq 1$, odavde dobijamo da je $1 \leq s - 1 - \sqrt{s^2 - 4q}$, što je ekvivalentno sa $s - 1 \leq q$.

Dokažimo sada desnu nejednakost u (a). Možemo pretpostaviti da je $s_1 < s_2$. Pokažimo da je

$$q(s_1, s_2, s_3, s_4, s_5) \leq q\left(s_2 + s_4 - \frac{1}{2}, s_1 + s_3 + s_5, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

Ova nejednakost je ekvivalentna sa

$$s_1 s_5 + \frac{s_2 + s_4}{2} \leq s_1 s_4 + s_2 s_5 + \frac{1}{4}.$$

Za fiksirane s_2, s_3, s_4, s_5 posmatrajmo na intervalu $(1/2, s_2)$ funkciju

$$g(t) = (s_4 - s_5)t + s_2 s_5 + \frac{1}{4} - \frac{s_2 + s_4}{2}.$$

Kako je g linearna funkcija iz

$$g\left(\frac{1}{2}\right) = \left(s_2 - \frac{1}{2}\right)\left(s_5 - \frac{1}{2}\right) \geq 0 \quad \text{i} \quad g\left(s_2\right) = \left(s_2 - \frac{1}{2}\right)\left(s_4 - \frac{1}{2}\right) \geq 0$$

slijedi da je $g(t) \geq 0$ za $1/2 \leq t \leq s_2$.
Neka je $x = s_2 + s_4 - \frac{1}{2}$, $y = s_1 + s_3 + s_5 - 1$. Tada je $x + y = s - \frac{3}{2}$, $x \geq \frac{1}{2}$ i $y \geq \frac{1}{2}$ pa imamo

$$q \leq q\left(x, y, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = xy + \frac{x+y}{2} + \frac{1}{2} \leq \left(\frac{x+y}{2}\right)^2 + \frac{x+y}{2} + \frac{1}{2} =$$

$$= \left(\frac{s-3/2}{2}\right)^2 + \frac{s-3/2}{2} + \frac{1}{2} = \frac{4s^2-4s+5}{16},$$

čime je i desna nejednakost dokazana.

Neka je $q = \frac{4s^2-4s+5}{16}$. Tada iz prethodno dokazanog slijedi da je $s_4 = s_5 = 1/2$ i $x = y$, tj. $s_2 + s_4 - 1/2 = s_1 + s_3 + s_5 - 1$. Uzimajući da je $s_1 = k/2$ i $s_3 = l/2$ dobijamo petorku

$$(s_1, s_2, s_3, s_4, s_5) = \left(\frac{k}{2}, \frac{k+l-1}{2}, \frac{l}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

kod koje je

$$q = \frac{(k+l)^2+1}{4} = \frac{4s^2-4s+5}{16} \quad \text{i} \quad 4(s^2-4q) = 4(k+l)-3.$$

Kako $4(s^2-4q)$ mora biti potpun kvadrat imamo da je $4(k+l)-3 = (2m+1)^2$, tj. $k+l = m^2+m+1$, $m \in \mathbb{N}$. Za svako $m \in \mathbb{N}$ postoji konveksan cjelobrojan petougao sa ovakvom petorkom $(s_1, s_2, s_3, s_4, s_5)$.
Zaista, iz $s_4 = 1/2 = s_5$, na osnovu posljedice 1 slijedi da možemo uzeti da je $A_3(0,1)$, $A_4(0,0)$, $A_5(1,0)$ i $A_1(m+1,1)$. Kako je $s_3 = l/2$ i

$$P_{A_2A_4A_5} = P - s_1 - s_3 = \frac{m^2+m+1}{2} - \frac{k}{2} - \frac{l}{2} = \frac{m+1}{2},$$

slijedi da je $A_2(l, m+1)$ (slika 5).

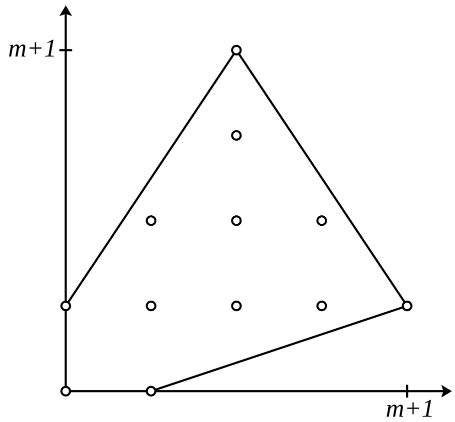
Dakle, za svaki konveksan cjelobrojan petougao kod koga je $q = \frac{4s^2-4s+5}{16}$ postoje prirodni brojevi m, l takvi da je on cjelobrojno ekvivalentan sa petouglom čija su tjemena $A_1(m+1,1)$, $A_2(l,m+1)$, $A_3(0,1)$, $A_4(0,0)$, $A_5(1,0)$.

Neka je $s - 1 = q = \frac{4s^2-4s+5}{16}$. Tada iz $s - 1 = \frac{4s^2-4s+5}{16}$ slijedi da je $s = 7/2$, pa na osnovu teoreme 3 dobijamo da je petougao X cjelobrojno ekvivalentan sa petouglom na slici 1. □

Iz prethodnog stava i posljedice 2 slijedi sljedeća teorema.

Teorema 4. *U konveksnom cjelobrojnom petouglu X važe sljedeće tvrdnje*

(a) $\frac{s+\sqrt{s-5/4}}{2} \leq P \leq s - 1.$



Slika 5:

(b) *Jednakost* $P = s - 1$ *važi ako i samo ako postoji prirodan broj* k *takav da je petougao* X *cjelobrojno ekvivalentan sa petouglom čija su tjemena* $A_1(1,0)$, $A_2(k+1,2)$, $A_3(k,2)$, $A_4(0,1)$, $A_5(0,0)$.

(c) *Jednakost* $P = \dfrac{s + \sqrt{s - 5/4}}{2}$ *važi ako i samo ako postoje prirodni brojevi* m, l *takvi da je petougao* X *cjelobrojno ekvivalentan sa petouglom čija su tjemena* $A_1(m+1,1)$, $A_2(l,m+1)$, $A_3(0,1)$, $A_4(0,0)$, $A_5(1,0)$.

(d) *Jednakosti* $\dfrac{s + \sqrt{s - 5/4}}{2} = P = s - 1$ *važe ako i samo ako je petougao* X *cjelobrojno ekvivalentan sa petouglom na slici 1.*

Provjerimo na kraju analitički dobijene rezultate na skupu konveksnih cjelobrojnih petouglova (sa tri parametra) čija su tjemena $A(0,1)$, $B(0,0)$, $C(1,0)$, $D(k,1)$, $E(m,n)$. Kako je on konveksan slijedi da su k, m, n prirodni brojevi veći od 1 i da je $m \leq n(k-1)$. Dalje imamo

$$s_1 = P_{ABC} = \frac{1}{2}\,,\ s_2 = P_{BCD} = \frac{1}{2}\,,\ s_3 = P_{CDE} = \frac{kn - m - n + 1}{2}\,,$$

$$s_4 = P_{DEA} = \frac{k(n-1)}{2}\,,\ s_5 = P_{EAB} = \frac{m}{2}\,,$$

odakle slijedi da je

$$s = s_1 + s_2 + s_3 + s_4 + s_5 = \frac{2kn - k - n + 3}{2}\,,\ P = \frac{kn+1}{2}\,,$$

$$q = s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_5 + s_5 s_1 = \frac{k^2n^2 - k^2n - kn^2 + 3kn - k - n + 2}{4}\,.$$

Zamjenom ovih vrijednosti za s i q u nejednakosti $s - 1 \leq q$, nakon sređivanja dobijamo da je ova nejadnakost ekvivalentna sa $(kn-1)(kn-k-n) \geq 0$. Kako su k i n prirodni brojevi veći od 1, ova nejednakost važi. Pri tome jednakost $s-1=q$ važi samo ako je $kn-k-n=0$, tj. $(k-1)(n-1)=1$, odakle dobijamo da je $k=n=2$. Dalje, iz uslova $m \leq n(k-1)$ slijedi da je $m \leq 2$, pa je $m=2$. Dakle, jednakost važi ako je petougao $ABCDE$ podudaran sa petouglom na slici 1.

Zamjenom ovih vrijednosti za s i q u nejednakosti $q \leq (4s^2-4s+5)/16$, nakon sređivanja dobijamo da je ova nejadnakost ekvivalentna sa $(k-n)^2 \geq 0$, što znači da vrijedi. Pri tome se jednakost $q=(4s^2-4s+5)/16$ dostiže samo za $k=n \geq 2$, uz uslov da je $m \leq k(k-1)$. Skup svih ovakvih petouglova $ABCDE$ se poklapa sa skupom petouglova na slici 3.

4 Zahvalnica

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Diofantove jednačine i parketiranje rani polupravilnim poligonima jedne vrste

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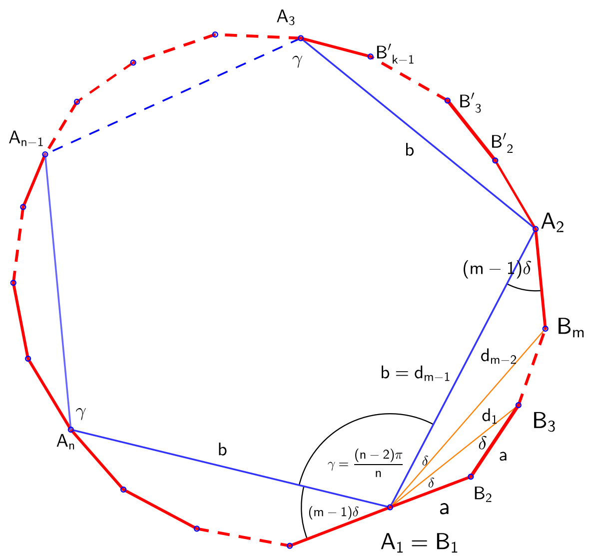
Abstrakt

U radu je razmatran problem parketiranja ravni konveksnim polupravilnim jednakostraničnim poligonima jedne vrste i odgovarajućih linearnih Diofantovih jednačina. Rješavanjem Dofantove jednačine koja odgovara parketiranju ravni polupravilnim poligonima jedne vrste, pokazano je da se od svih polupravilnih jednakostraničnih konveksnih poligona $\mathcal{P}_{2m}, m \geq 2$, parketiranje ravni može obaviti jedino sa polupravilnim četvorouglovima i šestouglovima. Dato je i nekoliko primjera parketiranja ravni konveksnim polupravilnim četvorouglovima i šestouglovima.

1 Uvod

Problem parketiranja ravni je drevni problem koji su razmatrali matematičari starog Egipta, Grčke, Persije, Kine i drugih starih civilizacija. Parketiranje se svodi na to da se ravan podijeli na poligone koji bi je potpuno prekrili a da pri tome nema preklapanja niti praznina uz određenu pravilnost obzirom na vrstu, oblik i raspored poligona. Dakle, kod parketiranja cilj je podijeliti ravan na poligone koji bi imali samo zajedničke stranice i vrhove. Tada za poligone koji imaju jednu stranicu zajedničku kažemo da su susjedni poligoni, a tačku ravni u kojoj se sastaju vrhovi susjednih poligona nazivamo čvorom te particije ravni. Čvorište nazivamo pravilnim ako su svi uglovi poligona koji se sastaju u njemu jednaki. Smatramo da su dva čvorišta jednaka ako je broj uglova koji se u njemu sastaju jednak. O problemu parketiranja se može naći u [1], [4], a katalog parketiranja se može vidjeti u [2],[3], [1].

Nas interesuju specijalni slučajevi parketiranja i to parketiranja ravni polupravilnim poligonima, odnosno kada se u čvorištu sastaju polupravilni poligoni jedne ili dvije i više vrsta.



Slika 1: Konveksni polupravilni jednakostranični poligon $\mathcal{P}_N$

Polupravilne poligone dijelimo u dvije grupe; *jednakostranične* (imaju jednake stranice,a različite uglove) i *jednakougaone* (imaju jednake unutrašnje uglove, a različite stranice). Svaka od tih grupa se može dalje podijeliti na konveksne i nekonveksne polupravilne poligone.

Za jednakostranični polupravilni poligon koji ima N stranica dužine a koristimo oznaku $\mathcal{P}_N^a$. Jedan takav polupravilni poligon prikazan je slikom (1).

Pravilni poligon $\mathcal{P}_n$ je "upisani",a jednakokraki poligon $\mathcal{P}_k$ je "ivični" poligon jednakostraničnom polupravilnom poligonu $\mathcal{P}_N$ sa $N = n \cdot m$ stranica.

Navedimo osnovne pojmove i oznake koji se odnose na tako konstruisane polupravilne konveksne poligone koje koristimo u daljem izlaganju zajedno sa njihovim osnovnim elementima.

1. Poligon $\mathcal{P}_k$ sa $m = k-1$ stranica konstruisan nad svakom stranicom A_jA_{j+1}, $j = 1, 2, \ldots, n$ poligona $\mathcal{P}_n$ sa kojim ima jednu stranicu zajedničku, nazivamo "ivični" poligon polupravilnog poligona $\mathcal{P}_N$.

2. $A_jB_2, B_2B_3, \ldots, B_{k-1}A_{j+1}, j = 1, 2, \ldots, n$ su stranice ivičnog poligona $\mathcal{P}_k$.

3. $A_jB_2, A_jB_3, \ldots, A_jB_{k-1}$ su dijagonale $d_i, i = 1, 2, \ldots, k-2$, ivičnog poligona $\mathcal{P}_k$ povučene iz zajedničkog vrha A_j i vrijedi

$$d_{k-2} = A_jA_{j+1} = b.$$

4. Unutrašnje uglove uz vrhove B_i poligona $\mathcal{P}_N$, označavamo sa β_i, a uz vrhove A_j sa α_j.

5. Sa δ označavamo ugao između dvije uzastopne dijagonale ivičnog poligona $\mathcal{P}_k$ povučene iz vrha $A_j, j = 1, 2, \ldots, n$ i vrijedi

$$\delta = \angle(a, d_1) = \angle(d_{i-1} d_i), i = 1, 2, \ldots, k-2 \tag{1.1}$$

i pri tome je $d_0 = a$ a $d_{k-2} = b$.

Neki od problema koji se mogu formulisati, u vezi tako konstruisanih polupravilnih poligona je i problem parketiranja ravni.

Prije nego razmotrimo problem parketiranja ravni tim poligonima, navedimo prvo jednu lemu, koja vrijedi za unutrašnje uglove polupravilnih jednakostraničnih poligona $\mathcal{P}_N$ konstruisanih na opisani način i jedan teorem koji je važan za njihovu konveksnost (za dokaz leme i teorema vidjeti [5],[6],[7]).

Lema koja vrijedi za unutrašnje uglove glasi

Lemma 1.1. *Polupravilni jednakostranični konveksni poligon $\mathcal{P}_N$ sa $N = n \cdot m$ stranica i uglom δ, ima n unutrašnjih uglova, uz vrhove "upisanog" pravilnog poligona $\mathcal{P}_n$, jednakih uglu α i vrijedi*

$$\alpha = \frac{(n-2)\pi}{n} + 2(m-1)\delta \tag{1.2}$$

i $(m-1) \cdot n$ unutrašnjih uglova,uz vrhove "ivičnih" jednakokrakih poligona $\mathcal{P}_k$, jednakih uglu β i vrijedi

$$\beta = \pi - 2\delta, \delta > 0, \quad m, n, k \in \mathbb{N}, m, n \geq 2, m = k-1. \tag{1.3}$$

Zavisnost konveksnosti polupravilnih jednakostraničnih poligona $\mathcal{P}_N$ od vrijednosti ugla δ iskazana je teoremom.

Teorema 1.2. *Jednakostranični polupravilni poligon $\mathcal{P}_N$ sa $N = n \cdot m$ stranica, konstruisan na opisani način je konveksan ako je*

$$\delta \in \left\langle 0; \tfrac{\pi}{N-n} \right\rangle; \delta \neq \frac{\pi}{N} \quad k, n, m \in \mathbb{N}, n, m \geq 2, m = k-1. \tag{1.4}$$

Primijetimo da je za $\delta = \frac{\pi}{N}$ konveksni polupravilni poligon pravilan, pa tu vrijednost ugla isključujemo iz razmatranja.

2 Parketiranje ravni polupravilnim poligonima jedne vrste

Parketiranje ravni jednakostraničnim konveksnim polupravilnim poligonima spada u posebnu grupu parketiranja. Na osnovu karakteristika, razlikujemo sljedeće vrste parketiranja ravni polupravilnim poligonima:

1. Parketiranje ravni polupravilnim poligonima kada se u svakom čvoru sastaje jednak broj polupravilnih poligona iste vrste,

2. Parketiranje ravni polupravilnim poligonima kada se u jednom čvoru sastaju polupravilni jednakostranični poligoni različitih vrsta i jednakih stranica,

3. Parketiranje ravni polupravilnim poligonima kada se u jednom čvoru sastaju polupravilni jednakostranični poligoni različitih vrsta i različitih stranica.

Ovdje razmatramo parketiranje ravni iz prvog slučaja. Ako je moguće parketirati ravan sa jednom vrstom polupravilnih poligona tada zbog postojanja dvije vrste unutrašnjih uglova razlikujemo sljedeće vrste čvorova:

1. Čvorovi u kojima se sastaju vrhovi kojima odgovaraju unutrašnji uglovi jednaki uglu α,

2. Čvorovi u kojima se sastaju vrhovi kojima odgovaraju unutrašnji uglovi jednaki uglu β,

3. Čvorovi u kojima se sastaju vrhovi kojima odgovaraju unutrašnji uglovi jednaki uglu α i uglu β.

Pretpostavimo da je moguće parketiranje ravni polupravilnim jednakostraničnim konveksnim poligonom iste vrste, konstruisanim na opisani način sa odgovarajućim karakterističnim elementima n, m, δ i unutrašnjim uglovima α, β.

Tada postoje nenegativni cijeli brojevi t, s koji nisu istovremeno jednaki nuli, takvi da se u jednom čvoru sastaje t vrhova kojima odgovaraju unutrašnji uglovi jednaki uglu α i/ili s vrhova kojima odgovaraju unutrašnji uglovi jednaki uglu β.

Na osnovu vrijednosti unutrašnjih uglova α i β i činjenice da je tada zbir uglova u svakom čvoru jednak 2π, navedeni uslovi se mogu zapisati u obliku jednačine

$$t \cdot \alpha + s \cdot \beta = 2\pi \quad \Leftrightarrow$$

$$t \cdot \Big[\frac{(n-2)\pi}{n} + 2(m-1)\delta\Big] + s \cdot (\pi - 2\delta) = 2\pi \tag{2.1}$$

Jednačina (2.1) je linearna Diofantova jednačina čije nepoznate $t, s \in \mathbb{Z}^+$, gdje je $\mathbb{Z}^+ = \{0\} \cup \mathbb{N}$, nisu istovremeno jednake nuli.

Izaberimo ugao $\delta = f(m, n)$ iz intervala u kojem je polupravilni jednakostranični poligon $\mathcal{P}_N, N = n \cdot m$ konveksan i koji se može geometrijski konstruisati odnosno,$\delta \in \langle 0; \frac{\pi}{n(m-1)} \rangle$, i $\delta \neq \frac{\pi}{N}$. Neka je δ oblika

$$\delta(n, m) = \frac{\pi}{2^m \cdot n \cdot (m-1)}, \quad n, m \geq 2 \in \mathbb{N}. \tag{2.2}$$

Zamijenimo li vrijednosti za uglove α, β i δ u jednačinu (2.1) dobijamo

$$t \cdot \Big[\frac{(n-2)\pi}{n} + \frac{2(m-1)\pi}{2^m n(m-1)}\Big] + s \cdot \Big(\pi - \frac{2\pi}{2^m n(m-1)}\Big) \quad = \quad 2\pi$$

$$\Leftrightarrow t\cdot\Big[\frac{(n-2)}{n}+\frac{2}{2^m n}\Big]+s\cdot\Big(1-\frac{2}{2^m n(m-1)}\Big) \;=\; 2$$
$$\Leftrightarrow t\cdot\Big(\frac{2^m(n-2)+2}{2^m n}\Big)+s\cdot\Big(\frac{2^m n(m-1)-2}{2^m n(m-1)}\Big) \;=\; 2.$$

Nakon sređivanja posljednja jednačina se može napisati u obliku

$$(m-1)(2^{m-1}(n-2)+1)\cdot t+(2^{m-1}n(m-1)-1)\cdot s=2^m n(m-1) \qquad (2.3)$$

koja predstavlja Diofantovu jednačinu u kojoj su nepoznate veličine t, s. Rješenje Diofantove jednačine je par $(t,s), t, s\in\mathbb{Z}^+$ gdje je $\mathbb{Z}^+=\{0\}\cup\mathbb{N}$ koji zadovoljava jednačinu (2.3). Označimo li cjelobrojne koeficijente redom sa: $A=(m-1)(2^{m-1}(n-2)+1), B=2^{m-1}n(m-1)-1, C=2^m n(m-1)$ jednačina se može napisati u jednostavnijem obliku

$$At+Bs=C.$$

Za različite vrijednosti n, m, jednačina ima različite oblike. Analizirajmo mogućnosti parketiranja ravni polupravilnim poligonima sa $N=2m, m\geq 2$, stranica, a ugao δ određen relacijom (2.2). U tom slučaju vrijedi teorema

Teorema 2.1. *Ravan možemo parketirati polupravilnim jednakostraničnim konveksnim poligonima $\mathcal{P}_{2m}, m\in\mathbb{N}$ jedino ako je $m=2$ i $m=3$. Za te vrijednosti m i ugao δ definisan sa (2.2) skup rješenja odgovarajuće Diofantove jednačine (2.3) je respektivno* $\{(8,0),(2,2)\}$ *i* $\{(16,0),(1,2)\}$.

Dokaz: Ako je $n=2$ Diofantova jednačina (2.3) prelazi u oblik

$$(m-1)\cdot t+(2^m(m-1)-1)\cdot s=2^{m+1}(m-1) \qquad (2.4)$$

Odredimo skup rješenja te jednačine u zavisnosti od $m\in\mathbb{N}$. Mogu se javiti sljedeći slučajevi:

1. Ako je $t\neq 0$ i $s=0$ jednačina (2.3) ima oblik

   $$(m-1)\cdot t=2^{m+1}(m-1) \qquad (2.5)$$

   Iz te jednačine nalazimo da je $t=2^{m+1}$. Odakle zaključujemo da kod parketiranja ravni polupravilnim poligonima $\mathcal{P}_{2m}, m\geq 2\in\mathbb{N}$ ima čvor $(t,s)=(2^{m+1},0)$ u kojem se nalazi 2^{m+1} uglova jednakih uglu α.

2. Ako je $t=0$ i $s\neq 0$ jednačina (2.3) ima oblik

   $$(2^m(m-1)-1)\cdot s=2^{m+1}(m-1) \qquad (2.6)$$

   Odakle je, $s=\frac{2^{m+1}(m-1)}{2^m(m-1)-1}$. Transformišimo izraz u oblik $s=2+\frac{2}{2^m(m-1)-1}$. Pa s je prirodan broj ako i samo ako je $\frac{2}{2^m(m-1)-1}$ prirodan broj. Odnosno, ako je $2^m(m-1)-1|2$. Ako bi to vrijedilo tada je $2^m(m-1)-1=\pm 1$ ili $2^m(m-1)-1=2$ jer je $s\neq 0$. U slučaju da je $2^m(m-1)-1=-1$ tada bi $2^m(m-1)=0$, što je moguće jedino za $m=1$, a to je suprotno pretpostavci da je $m\geq 2$. U slučaju da je $2^m(m-1)-1=1$ tada je $2^m(m-1)=2$. To je nemoguće, jer je za $m\geq 2$. Ako bi bilo $2^m(m-1)-1=2$ tada bi $2^m(m-1)=3$, odakle, slijedi da je $2^m=\frac{3}{m-1}$ što je nemoguće. Dakle, ne postoji ni jedan prirodan broj $m\geq 2$ za koji je izraz $\frac{2}{2^m(m-1)-1}$ prirodan broj, odnosno, s nije prirodan broj ni za jednu vrijednost $m\geq 2\in\mathbb{N}$. Na osnovu toga zaključujemo da ne postoji čvor u kojem se sastaju samo unutrašnji uglovi jednaki uglu β.

3. Ako su $t,s\neq 0$ jednačinu (2.4) transformišimo na sljedeći način:

$$\begin{aligned}
(m-1)\cdot t+(2^m(m-1)-1)\cdot s &= 2^{m+1}(m-1)\\
s &= \frac{2^{m+1}(m-1)-(m-1)\cdot t}{2^m(m-1)-1}\\
s &= \frac{2(2^m(m-1)-1)+2}{(2^m(m-1)-1)}-\frac{(m-1)\cdot t}{2^m(m-1)-1}\\
s &= 2+\frac{2-(m-1)\cdot t}{2^m(m-1)-1}.
\end{aligned}$$

Iz posljednje jednakosti slijedi da je

$$s=2+\frac{2-(m-1)\cdot t}{2^m(m-1)-1} \tag{2.7}$$

Očito će s biti prirodan broj

1. ako je $2-(m-1)\cdot t=0$, tada je $s=2$ ili
2. ako je $\frac{2-(m-1)\cdot t}{2^m(m-1)-1}=p$ prirodan broj.

1. Ako bi $2-(m-1)\cdot t=0$, tada je $t=\frac{2}{m-1}$ prirodan broj jedino za $m=2$ i $m=3$. U slučaju da je $m=2$ tada je $t=2$, i $s=2$, pa kod parketiranja ravni sa polupravilnim četverouglovima osim čvorova $(t,s)=(2^{m+1},0)=(8,0)$ imamo i čvorove $(t,s)=(2,2)$ u kojima se sastaju po dva unutrašnja ugla jednaki uglu α i uglu β. Ako je $m=3$ tada je $t=1$ a $s=2$ pa, parketiranje ravni sa polupravilnim šestouglovima osim čvora $(t,s)=(2^{m+1},0)=(16,0)$ ima i čvor $(t,s)=(1,2)$ u kojem se sastaju po jedan unutrašnji ugao jednaki uglu α i dva ugla jednaka uglu β.

2. Pretpostavimo da je s prirodan broj definisan jednakošću (2.7) i da je pri tome $\frac{2-(m-1)\cdot t}{2^m(m-1)-1}=p$ prirodan broj tj. $p\in\mathbb{N}$. Iz te jednakosti nalazimo da je

$$2-(m-1)t = p(2^m(m-1)-1)$$

$$
\begin{aligned}
(m-1)t &= 2+p-2^m(m-1)p \\
t &= \frac{p+2}{m-1} - 2^m \cdot p \\
t &= p(\frac{1}{m-1} - 2^m) + \frac{2}{m-1}
\end{aligned}
$$

Iz posljednje jednakosti slijedi da je t prirodan broj ako i samo ako je $\frac{1}{m-1} \in \mathbb{N}$ i ako je $\frac{2}{m-1} \in \mathbb{N}$. Uslovi su ispunjeni jedino ako je $m = 2$. No tada je $t = 2 - 3p, p \in \mathbb{N}$. Kako je $t > 0$ slijedi da mora biti $2 - 3p > 0$ odnosno $p < \frac{2}{3}$ što je suprotno pretpostavci da je p prirodan broj. Dakle, ne postoji prirodan broj $m \geq 2$ za koji je t prirodan broj, odnosno za koje je s definisan (2.7) prirodan broj. Na osnovu provedenog slijedi da su $s, t \neq 0$ prirodni brojevi jedino u slučaju kada je $2 - (m-1) \cdot t = 0$, i tada je $t = s = 2$, a $m = 2$ i $m = 3$.

Pokažimo na primjeru parketiranje ravni polupravilnim jednakostraničnim četvorouglovima i šestouglovima.

PRIMJER 2.1. Za polupravilni jednakostranični četvorougao $\mathcal{P}_4$ je $m = 2$ i $n = 2$, a ugao $\delta = \frac{\pi}{8}$ je određen relacijom $\delta(n,m) = \frac{\pi}{2^m n(m-1)}$. Odgovarajuća Diofantova jednačina

$$(m-1)t + (2^m(m-1)-1)s = 2^{m+1}(m-1)$$

u ovom slučaju glasi

$$t + 3s = 8,$$

gdje je t broj uglova jednakih unutrašnjem uglu α, a s broj uglova jednakih unutrašnjem uglu β koji se sastaju u jednom čvoru. Skup rješenja jednačine čine parovi $(t,s) = (2,2)$ i $(t,s) = (8,0)$ (Slika 2). Otuda zaključujemo da kod parketiranja ravni konveksnim polupravilnim četvorouglovima sa uglom $\delta = \frac{\pi}{8}$ imamo dvije vrste čvorova i to čvor u kojem se sastaje 8 vrhova sa unutrašnjim uglom α i one čvorove u kojima se sastaju 2 ugla jednaka uglu α i 2 ugla jednaka unutrašnjem uglu β.

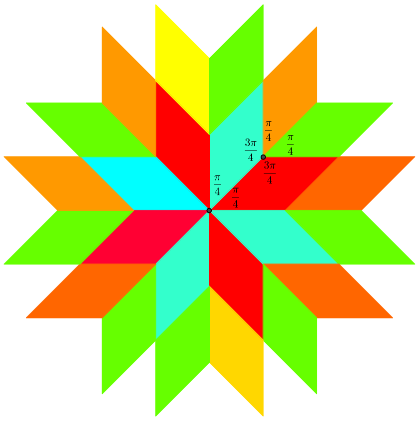
PRIMJER 2.2. Kako je za polupravilni šestougao $m = 3$ i $n = 2$, a ugao ugao $\delta = \frac{\pi}{32}$, odgovarajuća Diofantova jednačina glasi $2t + 15s = 32$. Skup riješenja te jednačine čine parovi $(t,s) = (1,2)$ i $(t,s) = (16,0)$.

Fragment parketiranja polupravilnim jednakostranim četvorouglovima ako je $\delta = \frac{\pi}{6}$ ima čvor tipa $(6,0)$ u kojem se sastaje šest uglova jednakih uglu $\alpha = \frac{\pi}{3}$ i čvor tipa (1,2) u kojem se sastaju jedan ugao jednak uglu $\alpha = \frac{\pi}{3}$ i dva ugla jednaka uglu $\beta = \frac{2\pi}{3}$(Slika3), kao i fragment parketiranja polupravilnim jednakostraničnim četvorouglovima sa uglom $\delta = \frac{\pi}{8}$ i čvorovima tipa $(2,2)$ prikazan je slikom 4.

Pokažimo prethodnu teoremu za polupravilne jednakostranične poligone $\mathcal{P}_N$, sa $N = n \cdot m$ stranica. Odnosno, pokažimo da Diofantova jednačina

$$(m-1)(2^{m-1}(n-2)+1) \cdot t + (2^{m-1}n(m-1)-1) \cdot s = 2^m n(m-1)$$

ima jedino rješenja ako je $n = 2$ i $m = 2$ ili $m = 3$.



Slika 2: Fragment parketiranja polupravilnim četvorouglom slučaj (8,0) i (2,2), $\delta = \frac{\pi}{8}$.

Teorema 2.2. *Parketiranje ravni konveksnim jednakostraničnim polupravilnim poligonima* $\mathcal{P}_N$, *sa* $N = n \cdot m, \quad n, m \geq 2, n, m \in \mathbb{N}$ *stranica, jedino je moguće sa polupravilnim četvorouglovima i šestouglovima.*

Dokaz: Ako je moguće parketiranje ravni sa polupravilnim jednakostraničnim poligonom $\mathcal{P}_N$ tada jednačina (2.3) ima cjelobrojna rješenja (t, s) za sve vrijednosti $n, m \geq 2 \in \mathbb{N}$ koji predstavljaju čvorove tog parketiranja. Pokažimo da je to moguće jedino ako je $n = 2$ a $m = 2$ ili $m = 3$.

Riješimo jednačinu u slučaju

1. ako je $s = 0$, a $t \neq 0$,

2. ako je $t = 0$, a $s \neq 0$, i

3. ako je $s, t \neq 0$.

1. Ako je $s = 0$, a $t \neq 0$ jednačina (2.3) ima oblik

$$(m-1)(2^{m-1}(n-2)+1) \cdot t = 2^m n(m-1),$$

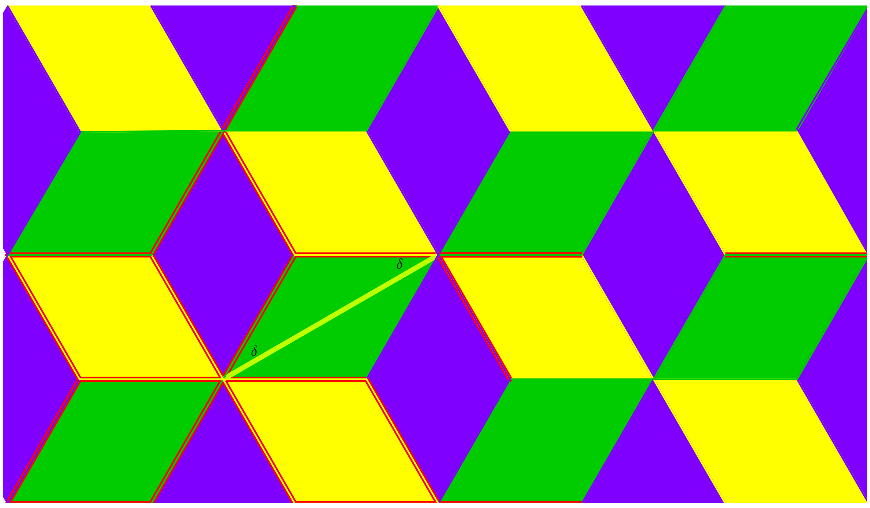
odakle nalazimo da je

$$t = \frac{2^m \cdot n}{2^{m-1}(n-2)+1}. \tag{2.8}$$

Iz jednakosti (2.8) slijedi da je $t \in \mathbb{N}$ ako i samo ako $2^{m-1}(n-2)+1 | 2^m \cdot n$. Transformišimo izraz u oblik

$$t = \frac{2^m \cdot n}{2^{m-1}(n-2)+1}$$

$$t = \frac{2^m \cdot n - 2^{m+1} + 2^{m+1}}{2^{m-1}(n-2)+1}$$



Slika 3: Fragment parketiranja polupravilnim četvorouglom za $\delta = \frac{\pi}{6}$.

$$\begin{aligned} t &= \frac{2(2^{m-1}(n-2)+1)+2^{m+1}-2}{2^{m-1}(n-2)+1} \\ t &= 2\cdot\frac{2^{m-1}(n-2)+1}{2^{m-1}(n-2)+1}+2\cdot\frac{2^m-1}{2^{m-1}(n-2)+1} \\ t &= 2+2\cdot\frac{2^m-1}{2^{m-1}(n-2)+1} \end{aligned}$$

Iz posljednje jednakosti slijedi da je $t \in \mathbb{N}$ ako i samo ako je $2^{m-1}(n-2)+1 | 2^m-1$, što je moguće jedino ako je $n-2=0$ odnosno ako je $n=2$, jer je $2^{m-1}(n-2)+1 = 2^{m-1}\cdot n-(2^m-1)$. Za tu vrijednost n imamo da je $t = 2+2(2^m-1) = 2^{m+1}$. Dakle, ako je $s=0$ tada je $t=2^{m+1}$, pa se kod parketiranja javlja čvor u kojem se sastaje 2^{m+1} uglova jednakih uglu α.

2. Ako je $t=0$, a $s \neq 0$, jednačina ima oblik

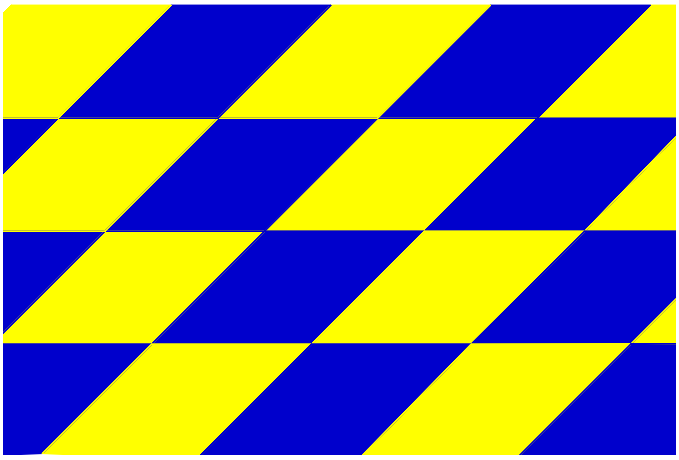
$$(2^{m-1}n(m-1)-1)\cdot s = 2^m n(m-1).$$

Odavde nalazimo da je

$$\begin{aligned} s &= \frac{2^m n(m-1)}{2^{m-1}n(m-1)-1} \\ &= 2\frac{2^{m-1}n(m-1)-1+1}{2^{m-1}n(m-1)-1} \\ &= 2\frac{2^{m-1}n(m-1)-1}{2^{m-1}n(m-1)-1}+\frac{2}{2^{m-1}n(m-1)-1} \\ &= 2+\frac{2}{2^{m-1}n(m-1)-1}. \end{aligned}$$

Iz jednakosti $s = 2 + \frac{2}{2^{m-1}n(m-1)-1}$ slijedi da je $s \in \mathbb{N}$ ako $2^{m-1}n(m-1) - 1|2$. Odnosno, ako je $2^{m-1}n(m-1) - 1 = \pm 1$ ili $2^{m-1}n(m-1) - 1 = \pm 2$.
Ako bi $2^{m-1}n(m-1) - 1 = -1$ tada bi $2^{m-1}n(m-1) = 0$ što nije moguće jer je po pretpostavci $n, m \geq 2 \in \mathbb{N}$.
Ako je $2^{m-1}n(m-1) - 1 = 1$ tada je $2^{m-1}n(m-1) = 2$. Kako za sve $n, m \geq 2 \in \mathbb{N}$ vrijedi $2^{m-1}n(m-1) > 2$ jednakost nije moguća. Ostaje da provjerimo jednakosti $2^{m-1}n(m-1) - 1 = 2$ i $2^{m-1}n(m-1) - 1 = -2$.
Ako je $2^{m-1}n(m-1) - 1 = 2$ tada je $2^{m-1}n(m-1) = 3$ što nije moguće ni za jedan prirodan broj $n, m \geq 2$.
Ako je $2^{m-1}n(m-1) - 1 = -2$ tada je $2^{m-1}n(m-1) = -1$ što nije moguće ni za jedan prirodan broj $n, m \geq 2$.

Na osnovu provedene analize zaključujemo da pri parketiranju ravni polupravilnim jednakostraničnim poligonom $\mathcal{P}_N$ nema čvorova u kojima je $t = 0$, a $s \neq 0$.



Slika 4: Fragment parketiranja polupravilnim četvorouglom za $\delta = \frac{\pi}{8}$.

3. Ostaje da analiziramo rješenja Diofantove jednačine ako je $t, s \neq 0$. Iz jednačine imamo da je

$$(m-1)(2^{m-1}(n-2)+1) \cdot t + (2^{m-1}n(m-1)-1) \cdot s = 2^m n(m-1), \quad \text{odakle je}$$
$$(2^{m-1}n(m-1)-1)s = 2^m n(m-1) - (m-1)(2^{m-1}(n-2)+1)t.$$

Odavde nalazimo da je

$$s = \frac{2^m n(m-1) - (m-1)(2^{m-1}(n-2)+1) \cdot t}{2^{m-1}n(m-1)-1}$$
$$s = \frac{2^m n(m-1)}{2^{m-1}n(m-1)-1} - \frac{(m-1)(2^{m-1}(n-2)+1) \cdot t}{2^{m-1}n(m-1)-1}$$
$$s = \frac{2(2^{m-1}n(m-1)-1}{2^{m-1}n(m-1)-1} + \frac{2-(m-1)(2^{m-1}(n-2)+1) \cdot t}{2^{m-1}n(m-1)-1}$$

$$s = 2+\frac{2-(m-1)(2^{m-1}(n-2)+1)\cdot t}{2^{m-1}n(m-1)-1}.$$

Analizirajmo jednakost

$$s=2+\frac{2-(m-1)(2^{m-1}(n-2)+1)\cdot t}{2^{m-1}n(m-1)-1}. \tag{2.9}$$

u zavisnosti od t, s Primjetimo da je $s \in \mathbb{N}$ ako je:

(A) $2-(m-1)(2^{m-1}(n-2)+1)\cdot t=0$ i tada je $s=2$, ili

(B) $\frac{2-(m-1)(2^{m-1}(n-2)+1)\cdot t}{2^{m-1}n(m-1)-1}=p, p\in\mathbb{N}$

Analizirajmo svaki od slučajeva. Pokažimo da jedino u slučaju (A)je $s \in \mathbb{N}$ i da je tada $s=2$, a da u slučaju (B) ne postoje prirodni brojevi $n, m \geq 2 \in \mathbb{N}$ za koje je taj razlomak prirodan broj.

Ako je $2-(m-1)(2^{m-1}(n-2)+1)\cdot t=0$ tada je $(m-1)(2^{m-1}(n-2)+1)\cdot t=2$ odakle je

$$t=\frac{2}{(m-1)(2^{m-1}(n-2)+1)} \tag{2.10}$$

Iz (2.10) slijedi da je $t\in\mathbb{N}$ ako i samo ako je $(m-1)(2^{m-1}(n-2)+1)=1$ ili $(m-1)(2^{m-1}(n-2)+1)=2$.

Ako je $(m-1)(2^{m-1}(n-2)+1)=1$ tada je $2^{m-1}(n-2)+1=\frac{1}{m-1}$, odnosno $2^{m-1}(n-2)=\frac{1-(m-1)}{m-1}$. Izraz na desnoj strani će biti prirodan broj ako i samo ako je $m=2$ i tada je desna strana jednaka nuli. Pa za tu vrijednost m jednakost vrijedi ako i samo ako je $n=2$. Dakle, ako je $n=2$ i $m=2$ tada je $t=2$. Na osnovu toga slijedi da u tom slučaju parketiranja ravni imamo čvor $(t,s)=(2,2)$ u kojima se sastaju po dva ugla jednaka uglu α i po dva ugla jednaka uglu β,i da se radi o parketiranju ravni sa polupravilnim jednakostraničnim četverouglovima.

Ako je $(m-1)(2^{m-1}(n-2)+1)=2$ tada je $2^{m-1}(n-2)+1=\frac{2}{m-1}$, odnosno $2^{m-1}(n-2)=\frac{2-(m-1)}{m-1}$. Primijetimo da je desna strana jednakosti $\frac{2-(m-1)}{m-1}$ prirodan broj ako i samo ako je $m\in\{2,3\}$.
Ako je $m=2$ tada iz jednakost $2^{2-1}\cdot(n-2)=1$ nalazimo da je $n=\frac{5}{2}\notin\mathbb{N}$, a u slučaju kada je $m=3$ nalazimo da je $n=2$.

Dakle, jedino zadovoljava drugi slučaj kada je $m=3$ u tom slučaju imamo da se radi o parketiranju polupravilnim šestouglovima, a čvorovi koji se javljaju su $(t,s)=(1,2)$.

Analizirajmo slučaj (B). Pokažimo da izraz $\frac{2-(m-1)(2^{m-1}(n-2)+1)\cdot t}{2^{m-1}n(m-1)-1}$ ne pripada skupu prirodnih brojeva ni za jednu vrijednost $n, m \geq 2 \in \mathbb{N}$. Pretpostavimo suprotno, da je

$$\frac{2-(m-1)(2^{m-1}(n-2)+1)\cdot t}{2^{m-1}n(m-1)-1}=p\in\mathbb{N}.$$

Iz te jednakosti nalazimo da je

$$t = \frac{2-(2^{m-1}n(m-1)-1)\cdot p}{(m-1)(2^{m-1}(n-2)+1)} \tag{2.11}$$

Pokažimo da t nije prirodan broj ni za jednu vrijednost $n, m \geq 2 \in \mathbb{N}$. Transformišimo izraz na sljedeći način

$$\begin{aligned} t &= \frac{2-(2^{m-1}n(m-1)-1)\cdot p}{(m-1)(2^{m-1}(n-2)+1)} \\ &= \frac{2}{(m-1)(2^{m-1}(n-2)+1)} - \frac{(2^{m-1}n(m-1)-1)\cdot p}{(m-1)(2^{m-1}(n-2)+1)}. \end{aligned}$$

Kako je $\frac{2}{(m-1)(2^{m-1}(n-2)+1)} \in \mathbb{N}$ ako je $(m-1)(2^{m-1}(n-2)+1) = 1$ ili $(m-1)(2^{m-1}(n-2)+1) = 2$.

Ako je $(m-1)(2^{m-1}(n-2)+1) = 1$ tada je $m = n = 2$, i iz $t = \frac{2-(2^{m-1}n(m-1)-1)\cdot p}{(m-1)(2^{m-1}(n-2)+1)}$ nalazimo da je $t = 2 - 3p$ gdje je p po pretpostavci prirodan broj. Da bi t bio prirodan broj mora biti $2-3p > 0$, odnonso $p < \frac{2}{3}$ što je suprotno pretpostavci da je p prirodan broj. U drugom slučaju, iz jednakosti $(m-1)(2^{m-1}(n-2)+1) = 2$ slijedi $2^{m-1}(n-2)+1 = \frac{2}{m-2}$. Desna strana je prirodan broj jedino ako je $m = 2$ ili $m = 3$. Ako je $m = 2$ tada iz jednakosti slijedi da je $n = \frac{5}{2}$, što je suprotno pretpostavci da je $n \geq 2 \in \mathbb{N}$, a u slučaju da je $m = 3$ je $n = 2$. Uvrstimo li to u jednakost (2.11) za t, nalazimo da je $t = 1 - \frac{7}{2}p$ što ne pripada skupu prirodnih brojeva ni za jedan prirodan broj p. Time smo dokazali da ni za jednu vrijednost prirodnih brojeva $n, m \geq 2$ izraz (2.11) nije prirodan broj. Time je teorem dokazan u potpunosti.

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Primena PID kontrolera necelobrojnog reda za stabilizaciju linearnih sistema upravljanja

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Abstract

Poslednjih decenija značajna pažnja naučne zajednice usmerena je na proučavanje linearnih frakcionih sistema, odnosno sistema opisanih linearnim diferencijalnim jednačinama necelobrojnog reda. Razlog za to je da se mnogi fizički sistemi mogu opisati diferencijalnim jednačinama ovog tipa, koje uključuju izvode necelobrojnog reda. PID kontroleri su, s druge strane, najzastupljeniji upravljački algoritmi u industriji, pre svega zbog svoje relativno jednostavne strukture i implementacije. Da bi se poboljšale performanse i robusnost klasičnog PID algoritma, uvodi se frakcioni PID ili $PI^{\lambda}D^{\mu}$ kontroler, gde su λ i μ integrator i diferencijator necelobrojnog reda, respektivno. Problem stabilnosti sistema je jedan od osnovnih zahteva u teoriji upravljanja. Postoji više pristupa za rešavanje ovog problema, medju kojima je i metoda D-razlaganja. U ovom radu, metoda D-razlaganja je uopštena i proširena za klasu linearnih diferencijalnih frakcionih jednačina, koje svoju primenu imaju u teoriji upravljanja. Razmatran je slučaj linearne zavisnosti parametara, a u konkretnom primeru opisan je i način na koji se izloženi postupak može iskoristiti i za rešavanje problema nelinearne parametarske zavisnosti. Prikazan je jednostavan i efikasan algoritam za odredjivanje granica stabilnosti u parametarskoj ravni. Testiranje ispravnosti predloženog algoritma izvršeno je u numeričkoj simulaciji u programskom paketu Matlab.

1 Uvod

Pri specifikaciji tehničkih zahteva za projektovanje sistema automatskog upravljanja prvo se mora uzeti u obzir najvažniji faktor, a to je stabilnost sistema. željeno dinamičko ponašanje objekta upravljanja može se ostvariti njegovom spregom sa upravljačkim uredjajima, koji svojim dejstvom treba da obezbedi zadovoljavajuće ponašanje celokupnog sistema. Sada se postavlja pitanje, kako treba izabrati podešljive parametre upravljačkog uredjaja da se ostvari postavljeni cilj. U opštem slučaju, zadato dinamičko ponašanje se može ostvariti u većem broju slučajeva i pri različitim kombinacijama vrednosti podešljivih parametara upravljačkog sistema, što znači da njihov izbor ne mora da bude jednoznačan.

Svakako da je prvi i osnovni zadatak da se obezbedi stabilan rad sistema upravljanja. Skup promenljivih parametara za koje je razmatrani sistem stabilan, čine oblast stabilnosti sistema. Upravo to čini suštinu metode D-razlaganja. Osnovna ideja te metode je da se odredi skup svih vrednosti podešljivih parametara za koje će razmatrani sistem biti stabilan. Time se u ravni podešljivih parametara dobijaju oblasti oivičene otvorenim ili zatvorenim konturama koje predstavljaju potencijalne oblasti stabilnosti [1]. Korišćenjem odgovarajućih postupaka utvrdjuje se kasnije koja od dobijenih oblasti, ukoliko postoji, predstavlja traženi skup podešljivih parametara za koji je sistem stabilan.

U ovom radu ćemo se ograničiti isključivo na linearne frakcione sisteme. Tačnije posmatrajmo stacionaran kontinualan frakcioni linearan sistem sa koncentrisanim parametrima. Dinamičko ponašanje takvog sistema se u potpunosti može okarakterisati diferencijalnom jednačinom ponašanja. Karakteristična jednačina takvog sistema glasi:

$$f(s) = a_n s^{b_n} + \ldots + a_k s^{b_k} + \ldots + a_0 s^{b_0} = 0 \tag{1}$$

gde je s kompleksna promenljiva, a_k i b_k $(k = 0, 1, 2 \ldots, n)$ linearne funkcije parametara α i β , pri čemu je $b_0 < b_1 < \ldots < b_n$.

2 Osnove računa necelobrojnog reda. Frakcioni PID kontroler

Za račun necelobrojnog reda (frakcioni račun) zna se već više od 300 godina, ali njegova primena u fizici i tehnici stara je tek nekoliko decenija. Sami koreni računa necelobrojnog reda vezuju se za korespondenciju koja je ostvarena izmedju Lopitala i Lajbnica, i to u vreme kad su Njutn i Lajbnic postavljali osnove diferencijalnog i integralnog računa. Račun necelobrojnog reda je generalizacija običnog (klasičnog) istoimenog računa [2, 3]. U matematičkom smislu, za razliku od klasičnog računa, ovde stepen može biti realan odnosno kompleksan broj, pa frakcioni račun ima potencijal da ostvari ono što obični integralno-diferencijalni račun ne može.

Tri definicije se najčešće koriste za račun necelobrojnog reda. Leva Riemann-Liouville definicija frakcionog izvoda je data sa [4]:

$${}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau, \tag{2}$$

za $n - 1 \leq \alpha < n$, gde je $\Gamma(.)$ dobro poznata gama funkcija:

$$\Gamma(z) = \int d^{-t} t^{z-1}\, dt, z \in C \tag{3}$$

Grunwald-ova definicija [5], pogodna za numeričko računanje,data je sa:

$${}^{GL}D_a^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{[t-a/h]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{4}$$

gde su a, t granice operatora, a $[x]$ je celobrojni deo od x. Izraz $\binom{\alpha}{j}$ predstavlja generalizovani binomijalni koeficijent, gde su faktorijali zamenjeni sa odgovarajućom gama funkcijom.

Takodje, koristi se i definicija levog frakcionog izvoda koju je uveo Caputo [6], a koja glasi:

$${}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau, \quad n-1 < \alpha < n \tag{5}$$

Caputo-va i Riemann-Liouville-va definicija frakcionog izvoda se podudaraju kada su početni uslovi jednaki nuli.

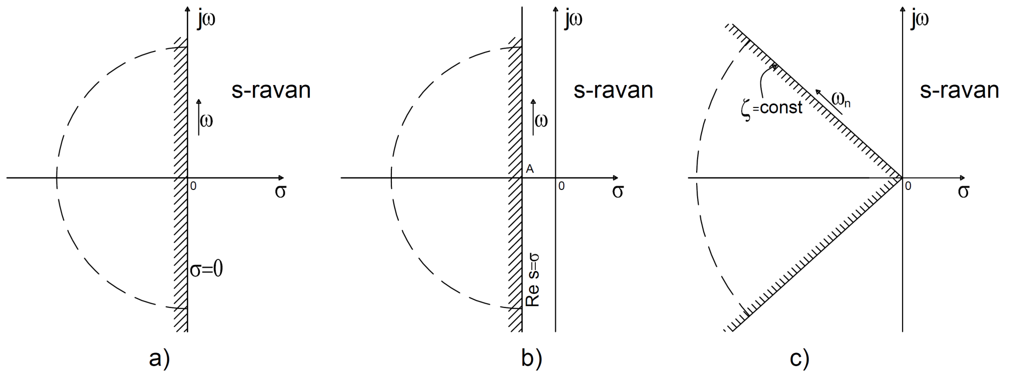
U teoriji upravljanja, cilj uvodjenja frakcionog računa je primena istoimenih kontrolera zarad poboljšanja performansi objekta upravljanja, tj. boljeg ponašanja sistema u prisustvu poremećajnih veličina i manje osetljivosti sistema na promenu parametara (veća robusnost). Frakcioni PID kontroler je generalizacija klasičnog (celobrojnog) PID kontrolera [7]. U literaturi je prisutna i oznaka $PI^{\lambda}D^{\mu}$ za ovu vrstu kontrolera zato što uključuje integrator i diferencijator necelobrojnog reda λ i μ , respektivno. Jednačina frakcionog PID kontrolera u vremenskom domenu je:

$$u(t) = K_P\, e(t) + K_I D^{-\lambda} e(t) + K_D D^{\mu} e(t) \tag{6}$$

gde je $u(t)$- izlaz iz kontolera, $e(t)$- ulaz u kontroler, K_P, K_I, K_D- koeficijenti proporcionalnog, integralnog i diferencijalnog pojačanja respektivno, a $D^{-\lambda}$ i D^{μ} odgovarajući integralni i diferencijalni operatori necelobrojnog reda. Za $\lambda = \mu = 1$ dobijamo jednačinu klasičnog PID upravljačkog algoritma. Upravo zbog dodatna dva podešljiva koeficijenta (λ i μ), frakcioni PID je fleksibilniji u odnosu na klasični, i daje mogućnost boljeg podešavanja dinamičkih osobina sistema. S druge strane, veći broj stepeni slobode čini problem optimalnog podešavanja parametara sistema znatno komplikovanijim u poredjenju sa konvencionalnim PID kontrolerom.

3 Stabilnost sistema

Potreban i dovoljan uslov za stabilnost sistema jeste svi koreni karakteristične jednačine (1) imaju negativne realne delove ili, drugim rečima, da leže u levoj poluravni kompleksne promenljive s (Slika 1a). Sistem će biti nestabilan ako se jedan ili više korena karakteristične jednačine nalaze u desnoj poluravni s-ravni ili ako jedan ili više višestrukih korena karakteristične jednačine leže na imaginarnoj osi s-ravni. Sistem će biti granično stabilan ako njegova karakteristična jednačina nema korena u desnoj poluravni, a pri tome ima bar jedan prost (jednostruk) koren na imaginarnoj osi [8].



Slika 1: Konture u s-ravni

U stabilnom sistemu, prelazni proces iščezava kada $t \to \infty$, a za to je potrebno i dovoljno da realni delovi svih korena budu negativni. Medjutim, evidentno je da će prelazni proces brže iščezavati ako su negativni delovi svih korena karakteristične jednačine veći po apsolutnoj vrednosti. Na taj način, od sistema se može zahtevati ne samo da bude stabilan, već se može specificirati zahtev da mu svi koreni karakteristične jednačine budu sa negativnim delovima, koji su po apsolutnoj vrednosti veći od nekog unapred zadatog $\sigma = \text{const.}$, odnosno da svi koreni karakteristične jednačine leže levo od prave $\mathrm{Re}\, s = \sigma$ u s-ravni, tj. unutar konture C na slici 1b. Za sistem koji ispunjava ovaj zahtev kažemo da poseduje vreme smirenja prelaznog procesa manje od nekog unapred zadatog.

Na sličan način mogu se specificirati i drugi domeni u s-ravni u kojima se zahteva da budu locirani svi koreni karakteristične jednačine sistema. Od posebnog interesa je domen pokazan na slici 1c. Specificiranjem ovog domena posebna pažnja se posvećuje lokaciji parova konjugovano kompleksnih korena karakteristične jednačine:

$$s_{1,2} = \sigma \pm j\omega = -\omega_n \varsigma \pm j\omega_n \sqrt{1-\varsigma^2} \tag{7}$$

Naime, zahteva se da svi kompleksni koreni krakteristične jednačine imaju odgovarajuće ς veće od nekog unapred zadatog $\varsigma = \text{const}$. Prisustvo kompleksnih korena u rešenju karakteristične jednačine uslovljava oscilatorni karakter komponenti prelaznog procesa sistema, stoga zahtev da svi koreni budu unutar domena na slici 1c u stvari predstavlja zadato ograničenje u pogledu maksimalno dozvoljenih amplituda oscilatornih komponenti prelaznog procesa u sistemu. Stoga za ove sisteme kažemo da imaju unapred zadati stepen relativne stabilnosti ograničen faktorom relativnog prigušenja $\varsigma = \text{const}$.

4 Metoda D razlaganja

Uopštavajući ranije postojeće rezultate, i dozvoljavajući da se dva podešljiva parametra α i β nadju u bilo kom koeficijentu karakteristične jednačine, ruski naučnik Neimark [9, 10] ustanovio je metodu D-razlaganja. Osnovna postavka te metode polazi od zahteva da se u parametarskoj ravni (α, β), odredi skup svih vrednosti podešljivih parametara za koje će razmatrani sistem, dat svojom karakterističnom jednačinom, biti stabilan. Znamo da će se se sistem nalaziti na granici stabilnosti ako njegova karakteristična jednačina nema korena sa pozitivnim realnim delovima, a ima jedan ili više jednostrukih korena na imaginarnoj osi s-ravni. Uslov da se sistem nadje na granici stabilnosti, tj. da jednačina (1) ima u rešenju jednostruke korene na imaginarnoj osi s-ravni, može se izraziti relacijom

$$f(j\omega) = u(\omega) + jv(\omega) = 0 \tag{8}$$

ili

$$u(\omega) = 0, \quad v(\omega) = 0 \tag{9}$$

Pretpostavimo da svi parametri sistema imaju konstantne vrednosti, osim dva, α i β, čiji uticaj na stabilnost sistema želimo da analiziramo [8]. Ova dva podešljiva parametra biće, na neki način, sadržana u koeficijentima karakteristične jednačine:

$$a_k = a_k(\alpha, \beta) \tag{10}$$

Ako u ovom slučaju smenimo $s = j\omega$ u karakterističnu jednačinu, dobićemo:

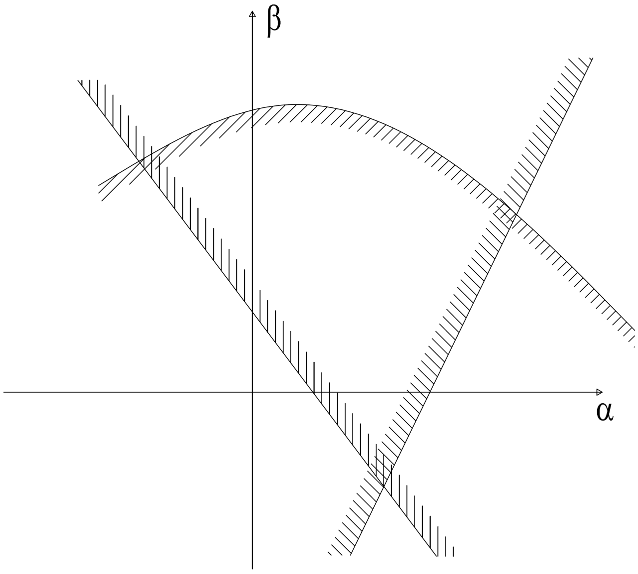
$$u(\omega, \alpha, \beta) = 0, \quad v(\omega, \alpha, \beta) = 0 \tag{11}$$

Ako su jednačine (11) medjusobno nezavisne, one se mogu rešiti po α i β:

$$\alpha = f_1(\omega), \quad \beta = f_2(\omega) \tag{12}$$

Zadavajući učestanosti ω razne vrednosti od 0 do ∞, u ravni parametara α i β se, pomoću relacija (12) može nacrtati familija krivih, koje se nazivaju krivama dekompozicije ili krivama razlaganja. One će u stvari predstavljati granicu domena stabilnosti iz s-ravni, preslikanu pomoću relacija (12) u ravan podešljivih parametara α i β. Pomenuta familija krivih dekomponuje parametarsku ravan na više oblasti. Svakoj od tih oblasti odgovaraće tačno odredjen broj korena karakteristične jednačine, koji se nalaze u levoj poluravni s-ravni.

Krive razlaganja se mogu razumeti i kao geometrijska mesta tačaka u (α, β) ravni duž kojih karakteristična jednačina sistema ima u rešenju korene na imaginarnoj osi s-ravni. Zbog toga, u oblasti sa jedne strane krive dekompozicije karakteristična jednačina će imati jedan ili dva korena više u levoj poluravni nego sa suprotne strane ove krive i to u zavisnosti od toga da li ta kriva predstavlja geometrijsko mesto jednog realnog ($\sigma = 0$) ili para konjugovano kompleksnih korena karakteristične jednačine na imaginarnoj osi s-ravni. Radi što lakšeg odredjivanja broja korena karakteristične jednačine sa negativnim realnim delovima, koji odgovara svakoj od pojedinih oblasti u parametarskoj (α, β) ravni, krive razlaganja se šrafiraju i to sa strane domena kome odgovara veći broj ovih korena, slika 2.



Slika 2: Krive razlaganja u (α, β) parametarskoj ravni

Oblast kojoj odgovara najveći broj korena karakteristične jednačine u levoj poluravni s-ravni, predstavlja tzv. potencijalnu oblast stabilnosti, tj. oblast koja pretenduje da bude oblast stabilnosti u (α, β) ravni. Da bi ona to zaista i bila, potrebno je dokazati da za jednu, proizvoljnu tačku unutar te oblasti karakteristična jednačina ima sve korene sa negativnim realnim delovima, tj. da je broj ovih korena, koji odgovara potencijalnoj oblasti jednak n, gde je n stepen karakteristične jednačine. ako se pokaže da je taj broj manji od n, to će značiti da ne postoji ni jedan par vrednosti parametar α i β za koji je posmatrani sistem stabilan. Drugim rečima, u odnosu na te parametre sistem je strukturno nestabilan.

Pri nanošenju šrafure potrebno je rukovoditi se sledećim pravilom, koje navodimo bez dokaza. Ako se po apscisnoj osi parametarske ravni nanosi parametar α, a po ordinati β, tada se šrafura nanosi u smeru zavisnom od znaka Jakobijana:

$$J = \begin{vmatrix} \frac{\partial u}{\partial \alpha} & \frac{\partial u}{\partial \beta} \\ \frac{\partial v}{\partial \alpha} & \frac{\partial v}{\partial \beta} \end{vmatrix} = \frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta} - \frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \alpha} \tag{13}$$

Ako je pri kretanju duž krive razlaganja u smeru porasta ω Jakobijan J pozitivan, kriva se šrafira sa leve strane, gledano u smeru porasta ω, pri negativnom J, s desne.

Utvrdjivanje da li potencijalna oblast stabilnosti predstavlja zaista oblast stabilnosti može se izvršiti primenom nekog od algebarskih (Raus, Hurvic), grafoanalitičkih (Najkvist) kriterijuma, ili numeričkom simulacijom, a za proizvoljno izabranu tačku unutar te potencijalne oblasti stabilnosti.

U daljem izlaganju posmatraće se problemi izdvajanja stabilne oblasti u slučaju kada se parametri α i β javljaju linearno u koeficijentima karakteristične jednačine. Tad relacije (10) postaju:

$$a_k = a_k(\alpha, \beta) = c_k\alpha + d_k\beta + e_k, \quad k = 0, 1, 2, \ldots, n \tag{14}$$

gde koeficijenti c_k, d_k, i e_k imaju konstantne vrednosti. Posle zamene (14) u karakterističnu jednačinu (1), dobija se:

$$f(s) = \alpha P(s) + \beta Q(s) + R(s) \tag{15}$$

gde su $P(s), Q(s)$ i $R(s)$ odgovarajući polinomi po s sa realnim koeficijentima. Postavljajući $s = j\omega$ u (15) i zatim izjednačavajući realni i imaginarni deo sa nulom, dobija se:

$$\begin{aligned} \alpha P_1(\omega) + \beta Q_1(\omega) + R_1(\omega) = 0 \\ \alpha P_2(\omega) + \beta Q_2(\omega) + R_2(\omega) = 0 \end{aligned} \tag{16}$$

gde su $P(j\omega) = P_1(\omega) + jP_2(\omega)$, $Q(j\omega) = Q_1(\omega) + jQ_2(\omega)$, i $R(j\omega) = R_1(\omega) + jR_2(\omega)$. Rešavajući jednačine (16), dobijamo izraze za α i β u obliku:

$$\begin{aligned} \alpha = \frac{Q_1(\omega)R_2(\omega) - Q_2(\omega)R_1(\omega)}{\Delta} \\ \beta = \frac{R_1(\omega)P_2(\omega) - R_2(\omega)P_1(\omega)}{\Delta} \end{aligned} \tag{17}$$

gde je

$$\Delta = \begin{vmatrix} P_1(\omega) & Q_1(\omega) \\ P_2(\omega) & Q_2(\omega) \end{vmatrix} = P_1(\omega)Q_2(\omega) - Q_1(\omega)P_2(\omega) \tag{18}$$

Pri tom su mogući sledeći slučajevi.

1. Determinanta Δ nije jednaka nuli i R_1 i R_2 nisu jednovremeno jednaki nuli. Jednačine (16) su linearno nezavisne i rešenje (17) za dato ω odredjuje odgovarajuću tačku u (α, β) ravni.

2. Za neku vrednost ω determinanta Δ je jednaka nuli, a brojioci u izrazima za α i β tada nisu jednovremeno jednaki nuli. Tada su jednačine (16) nesaglasne i nemaju konačno rešenje. Odgovarajuća tačka u (α, β) ravni se nalazi u beskonačnosti i nju nije moguće naneti na grafik.

3. Pri nekoj vrednosti ω i brojioci izraza za α i β i determinanta Δ jednovremeno postaju jednaki nuli, tj. parametri α i β postaju neodredjeni. U tom slučaju, kao što je poznato jednačine (16) postaju linearno zavisne, tj. jedna od njih postaje jednaka drugoj ako se pomnoži sa odgovarajućom konstantom. Ovoj vrednosti za u (α, β) ravni odgovara prava linija

$$\alpha P_1 + \beta Q_1 + R_1 = 0 \tag{19}$$

koja se naziva singularnom pravom [11]. Ona ne ulazi u familiju krivih dekompozicije, pošto svim tačkama te prave odgovara ista vrednost za ω, pa se kretanje duž prave u smeru porasta ω ne može odrediti. Uočimo da vrednosti $\omega = 0$ uvek odgovara singularna prava u (α, β) ravni. Naime, $P_2(\omega), Q_2(\omega), R_2(\omega)$ su neparne funkcije po ω, pa se ω uvek može izvući kao faktor u izrazima za ove funkcije: $P_2(\omega) = \omega P_{20}(\omega)$, $Q_2(\omega) = \omega Q_{20}(\omega)$, $R_2(\omega) = \omega R_{20}(\omega)$. Kada je $\omega = 0$, dobija se $\alpha = 0/0$, $\beta = 0/0$ tj. dobijamo singularnu pravu. Ova prava se neposredno može dobiti iz jednačina (16):

$$\begin{aligned} &\alpha P_1(0) + \beta Q_1(0) + R_1(0) = 0 \\ &\omega\left(\alpha P_{20}(0) + \beta Q_{20}(0) + R_{20}(0)\right) = 0 \end{aligned} \tag{20}$$

Nije teško uočiti da ova prava odgovara slobodnom članu karakteristične jednačine izjednačenim sa nulom: $a_0(\alpha, \beta) = b_0\alpha + c_0\beta + d_0 = 0$. Dakle, ova prava predstavlja geometrijsko mesto tačaka u (α, β) ravni duž koga karakteristična jednačina ima u rešenju jedan prost koren $s = 0$ u koordinatnom početku s-ravni.

Singularna prava se može pojaviti i za vrednost različite ω od nule. U (α, β) ravni takva jedna prava će predstavljati granicu vrednosti parametara α i β pri kojoj par kompleksnih korena karakteristične jednačine prelazi preko imaginarne ose iz jedne poluravni s-ravni u drugu.

Za vrednost $\omega = \infty$ takodje se može pojaviti singularna prava. Ovaj slučaj nastupa kad god se jedan od parametara α ili β ili oba pojavljuju u koeficijentu a_n najstarijeg člana karakteristične jednačine (1). Jednačina ove prave se može dobiti iz uslova $a_n(\alpha, \beta) = b_n\alpha + c_n\beta + d_n = 0$ pošto promena znaka za a_n narušava uslove stabilnosti sistema. Ova prava će predstavljati granicu u (α, β) ravni preko koje jedan koren karakteristične jednačine prelazi kroz beskonačnost iz jedne u drugu poluravan s-ravni.

Singularna prava se šrafira ukoliko ima bar jednu zajedničku tačku sa krivom razlaganja za istu vrednost učestanosti ω . Ako singularna prava asimptotski teži krivoj razlaganja onda se u beskonačnosti šrafura prenosi sa krive razlaganja na singularnu pravu. Ako pak u zajedničkoj tački za pripadnu vrednost učestanosti i determinanta Δ menja znak tada će u toj tački signularna promeniti stranu svoje šrafure. Singularna prava ne menja stranu svoje šrafure ako u tačkama preseka sa krivom razlaganja determinanta Δ ne menja znak [1].

5 Numerički primer i simulacija rezultata

Neka je data karakteristična jednačina sistema u sledećem obliku [12]:

$$f(s) = s^4 + k_3\alpha s^2 s^\mu + \beta s^2 s^\lambda + k_2 s^2 + k_1\alpha s^\mu + k_0 \tag{21}$$

gde su α i β podešljivi parametri sistema, λ i μ izvodi necelobrojnog reda, a $k_3, k_2, k_1, k_0 \neq 0$ konstantni koeficijenti. Podešljivi parametri sistema α i β , se pojavljuju linearno u koeficijentima karakteristične jednačine. Vrednost necelobrojnih izvoda λ i μ kreće se u opsegu od 0 do 2. Koristeći gore opisani postupak D razlaganja, odredimo oblast stabilnosti navedenog sistema u (α, β) ravni. Kao što je već rečeno, krive razlaganja predstavljaju geometrijsko mesto tačaka u (α, β) ravni za koje karakteristični polinom (21) ima nule na imaginarnoj osi. Zamenjujući $s = j\omega$ u (21) i izjednačavajući dobijeni izraz sa nulom, dobijamo:

$$f(j\omega) = \omega^4 - k_3\alpha\omega^2(j\omega)^\mu - \beta\omega^2(j\omega)^\lambda - k_2\omega^2 + k_1\alpha(j\omega)^\mu + k_0 \tag{22}$$

Gornja jednačina se može napisati kao:

$$f(j\omega) = u(\omega, \alpha, \beta) + jv(\omega, \alpha, \beta) \tag{23}$$

gde $u(\omega, \alpha, \beta)$ i $v(\omega, \alpha, \beta)$ predstavljaju realni i imaginarni deo jednačine (22). Izrazi $(j\omega)^\mu$ i $(j\omega)^\lambda$ koji se pojavljuju u istoj jednačini, mogu se izraziti kao

$$(j\omega)^\mu = \omega^\mu \left(\cos(0.5\mu\pi) + j\sin(0.5\mu\pi)\right), \quad \omega \geq 0 \tag{24}$$

Izjednačavajući realni i imaginarni deo jednačine (23) sa nulom, dobijamo sledeći sistem od dve jednačine:

$$\begin{bmatrix} U_1(\omega, \mu, \lambda) & U_2(\omega, \mu, \lambda) \\ V_1(\omega, \mu, \lambda) & V_2(\omega, \mu, \lambda) \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} Q_1(\omega) \\ Q_2(\omega) \end{pmatrix} \tag{25}$$

gde su

$$\begin{aligned} U_1(\omega, \mu, \lambda) &= (k_1 - k_3\omega^2)\,\omega^\mu \cos(0.5\mu\pi) \\ U_2(\omega, \mu, \lambda) &= -\omega^{2+\lambda}\cos(0.5\mu\pi) \\ V_1(\omega, \mu, \lambda) &= (k_1 - k_3\omega^2)\,\omega^\mu \sin(0.5\mu\pi) \\ V_2(\omega, \mu, \lambda) &= -\omega^{2+\lambda}\sin(0.5\mu\pi) \\ Q_1(\omega) &= -\omega^4 + k_2\omega^2 - k_0 \\ Q_2(\omega) &= 0 \end{aligned} \tag{26}$$

Rešavanjem (25) po α i β, dobija se:

$$\alpha = \frac{\Delta_\alpha}{\Delta}, \quad \beta = \frac{\Delta_\beta}{\Delta} \tag{27}$$

pri čemu su

$$\Delta = \begin{vmatrix} U_1(\omega, \mu, \lambda) & U_2(\omega, \mu, \lambda) \\ V_1(\omega, \mu, \lambda) & V_2(\omega, \mu, \lambda) \end{vmatrix} \tag{28}$$

$$\Delta_\alpha = \begin{vmatrix} Q_1(\omega) & U_2(\omega, \mu, \lambda) \\ 0 & V_2(\omega, \mu, \lambda) \end{vmatrix}, \quad \Delta_\beta = \begin{vmatrix} U_1(\omega, \mu, \lambda) & Q_1(\omega) \\ V_1(\omega, \mu, \lambda) & 0 \end{vmatrix} \tag{29}$$

Lako se pokazuje da je glavna determinanta sistema jednaka:

$$\Delta = (k_1 - k_3\omega^2)\,\omega^{\mu+\lambda+2}\sin(0.5(\mu - \lambda)\pi) \tag{30}$$

Računavajući pomoću jednačina (27) vrednosti za α i β pri raznim ω , a za $\Delta \neq 0$, dobijaju se krive razlaganja. Drugim rečima, pri prelasku ovih krivih, dva korena karakteristične jednačine sistema prelaze imaginarnu osu u s-ravni s jedne na drugu stranu.

Sada ćemo detaljnije analizirati slučaj kada je $\Delta = 0$. Iz (30) sledi da je ovo tačno za $\omega = 0$ ili $\mu - \lambda = i, \forall i = 0, \pm 2, \pm 4, \ldots$. U prvom slučaju kada je $\omega = 0$, jednačine (26) glase:

$$\begin{aligned} U_1(0,\mu,\lambda) &= 0 \quad U_2(0,\mu,\lambda) &= 0 \quad Q_1(0) = k_0 \\ V_1(0,\mu,\lambda) &= 0 \quad V_2(0,\mu,\lambda) &= 0 \quad Q_2(0) = 0 \end{aligned} \tag{31}$$

Iz (25) i (31) sledi $0 = -k_0$. Ovo ne može biti tačno za $k_0 \neq 0$, pa sledi da system (25) nema singularnu pravu kada je $\omega = 0$. U drugom slučaju, glavna determinanta Δ je jednaka nuli za $\mu - \lambda = i, \forall i = 0, \pm 2, \pm 4, \ldots$. S obzirom da se vrednost fracionih izvoda μ, λ kreće u intervalu $(0, 2)$, sledi da je $\mu - \lambda = 0$. Sada za $\mu = \lambda$, jednačine (26) glase:

$$\begin{aligned} U_1(\omega,\mu,\mu) &= (k_1 - k_3\omega^2)\,\omega^{\mu}\cos(0.5\mu\pi) \\ U_2(\omega,\mu,\mu) &= -\omega^{2+\mu}\cos(0.5\mu\pi) \\ V_1(\omega,\mu,\mu) &= (k_1 - k_3\omega^2)\,\omega^{\mu}\sin(0.5\mu\pi) \\ V_2(\omega,\mu,\mu) &= -\omega^{2+\mu}\sin(0.5\mu\pi) \\ Q_1(\omega) &= -\omega^4 + k_2\omega^2 - k_0 \\ Q_2(\omega) &= 0 \end{aligned} \tag{32}$$

Sistem jednačina (25) može se napisati kao:

$$\omega^{\mu}\cos(0.5\mu\pi)\left[(k_1 - k_3\omega^2)\alpha - \omega^2\beta\right] = -\omega^4 + k_2\omega^2 - k_0 \tag{33}$$

$$\omega^{\mu}\sin(0.5\mu\pi)\left[(k_1 - k_3\omega^2)\alpha - \omega^2\beta\right] = 0 \tag{34}$$

što vodi do (za $\mu = \lambda \neq 0$):

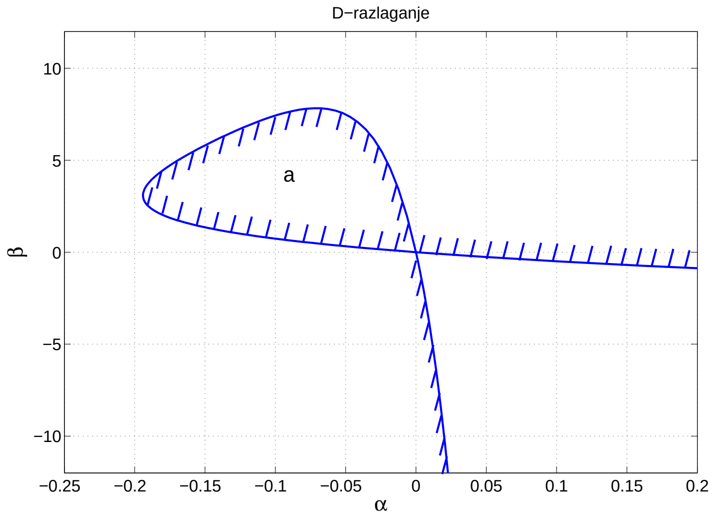
$$d(\omega) = -\omega^4 + k_2\omega^2 - k_0 = 0 \tag{35}$$

Frekvencija ω_s za koju važi $d(\omega_s) = 0$ odredjuje singularnu liniju. U ovom slučaju je $\Delta = \Delta_\alpha = \Delta_\beta = 0$, i u parametarskoj ravni nemamo samo jednu tačku, već pravu liniju. Ova singularna prava se može dobiti iz (33) ili (34):

$$\beta = \frac{k_1 - k_3\omega_s^2}{\omega_s^2}\alpha \tag{36}$$

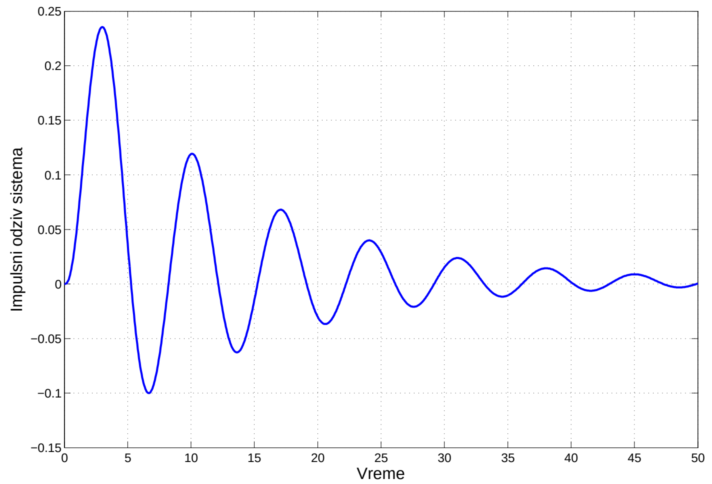
Iz jednačina (27) i (36) se odredjuju granice stabilnosti za sistem (22) u parametarskoj (α, β) ravni, a za fiksne vrednosti k_3, k_2, k_1, k_0, μ i λ.

Simulacija izvedenog algoritma za odredjivanje granica stabilnosti u parametarskoj (α, β) ravni izvršena je u Matlab programskom paketu. Za sledeće vrednosti koeficijenata $k_3 = 1.3$, $k_2 = 6$, $k_1 = -45$, $k_0 = 1$, $\mu = 0.7$ i $\lambda = 1.1$, sračunate su vrednosti parametara α i β pri raznim ω, i dobijena kriva razlaganja je prikazana na slici 3. Singularna prava u ovom slučaju ne postoji, jer je $\mu \neq \lambda$. Šrafiranje krive izvršeno je prema ranije navedenom pravilu.



Slika 3: Krive razlaganja u (α, β) parametarskoj ravniza sistem u primeru

Da bismo proverili da li je naša potencijalna oblast zaista stabilna, izaberimo proizvoljnu tačku a na slici 3. unutar te oblasti. Sada vrednosti (α, β) parametara dobijaju konkretne vrednosti $(\alpha = -0.1, \beta = 5)$. Za te vrednosti parametara izvršena je numerička simulacija impulsnog odziva sistema $1/f(s)$ u Matlab okruženju, i dobijeni odziv je prikazan na slici ispod. Vidimo da prelazni proces iščezava kada $t \to \infty$, što je u saglasnosti sa definicijom stabilnog sistema.



Slika 4: Impulsni odziv sistema $1/f(s)$

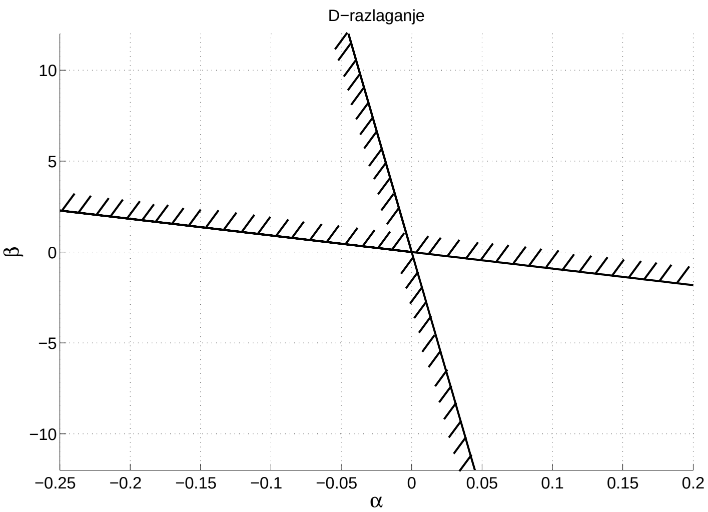
U slučaju da je $\mu = \lambda = 1.1$ (koeficijenti k_3, k_2, k_1, k_0 ne menjaju svoje vrednosti), parametarska ravan (α, β) odredjena je samo singularnim pravama, kao što je prikazano na slici 5. Tada je polinom (35) jednak nuli za $\omega^*_{1,2} = \pm 2.41$ i $\omega^*_{3,4} = \pm 0.41$, važi $\Delta = \Delta_\alpha = \Delta_\beta = 0$ i singularne prave imaju sledeći oblik:

$$\beta_1 = -9.11\alpha, \quad \beta_2 = -267.2\alpha \tag{37}$$

Kriva razlaganja u ovom slučaju ne postoji, jer jednačine (25) sada glase:

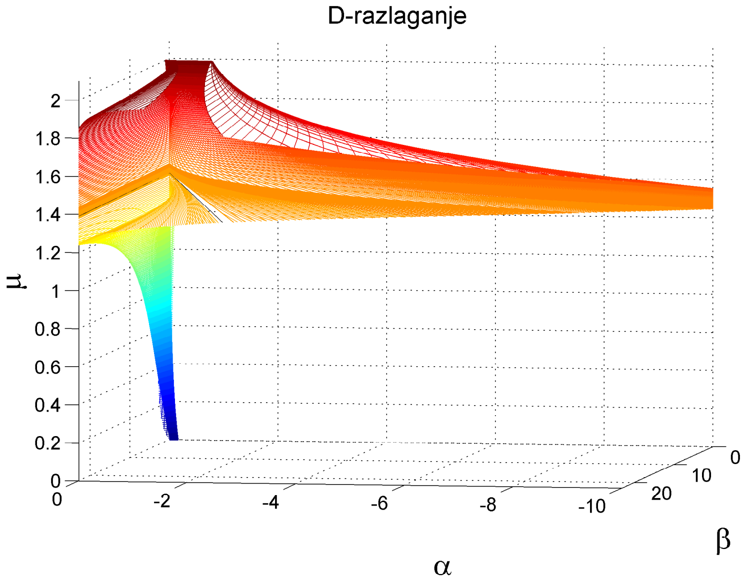
$$\begin{aligned} &\omega^{1.1}\cos(\frac{1.1\pi}{2})\left[(-45 - 1.3\omega^2)\alpha - \omega^2\beta\right] = -\omega^4 + 6\omega^2 - 1 \\ &\omega^{1.1}\sin(\frac{1.1\pi}{2})\left[(-45 - 1.3\omega^2)\alpha - \omega^2\beta\right] = 0 \end{aligned} \tag{38}$$

i one su nesaglasne za $\forall\omega \neq \{\pm 2.41, \pm 0.41\}$. Za $\omega = \omega^*$ jednačine (25) postaju linearno zavisne i odredjuju već pomenute singularne prave.



Slika 5: Singularne prave u (α, β) parametarskoj ravni za sluča $\mu = \lambda$

Ukoliko menjamo postepeno vrednost npr. frakcionog izvoda μ od 0 do 2, pri $\lambda = \text{const.}$, i u svakom koraku računamo granice stabilnosti u (α, β) parametarskoj ravni, možemo proširiti postojeće rezultate i dobiti oblast stabilnosti u trodimenzionalnoj (α, β, μ) parametarskom prostoru. Na taj način metoda D-razlaganja proširena je za slučaj nelinearne zavisnosti parametara, jer navedeni parametri su u medjusobno nelinearnoj vezi, što se vidi iz jednačina (25) i (26). Na slici 6. prikazana je oblast stabilnosti u 3D prostoru, za $\lambda = 1.4$ (koeficijenti k_3, k_2, k_1, k_0 su ostali nepromenjeni).



Slika 6: Oblast stabilnosti u (α, β, μ) parametarskom prostoru

6 Zaključak

U ovom radu analiziran je problem stabilnosti sistema opisanih linearnim diferencijalnim jednačinama necelobrojnog reda. Dat je kratak osvrt na osobine stabilnosti sistema sa stanovišta teorije upravljana. Takodje, dat je uvod u osnove računa necelobrojnog reda. Upotrebljena je metoda D-razlaganja za odredjivanje granica stabilnosti u ravni podešljivih parametara sistema. Ova metoda je uopštena i proširena za slučaj linearnih diferencijalnih frakcionih jednačina, što predstavlja glavni doprinos u ovom radu. Prikazan je jednostavan i efikasan algoritam odredjivanja granica stabilnosti za slučaj linearne zavisnosti parametara. Zatim je u konkretnom primeru opisan način na koji se dati algoritam može iskoristiti i za slučaj nelinearne parametarske zavisnosti. Testiranje ispravnosti predloženog algoritma izvršeno je u numeričkoj simulaciji u programskom paketu Matlab.

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